4.2 (2, 6, 8, 10, 12, 14, 16, 22, 26, 34, 40, 44, 48, 52, 58)

(2) $f(x) = x^{2/3}$, $[0, 1]$

$f(1) = 1$, $f(0) = 0$, $\frac{\Delta y}{\Delta x} = \frac{f(1) - f(0)}{1 - 0} = \frac{1-0}{1-0} = 1$

$f'(x) = \frac{2}{3} x^{-1/3}$

$f'(c) = \frac{2}{3} c^{-1/3} = 1$

$\frac{2}{3} = c^{1/3}$

$\boxed{\frac{8}{27} = c}$

(6) $f(x) = \ln(x-1)$, $[2, 4]$

$f(4) = \ln(3)$, $f(2) = \ln(1) = 0$ $\quad \frac{\Delta y}{\Delta x} = \frac{\ln(3) - 0}{4-2} = \frac{\ln(3)}{2}$

$f'(x) = \frac{1}{x-1}$

$f'(c) = \frac{1}{c-1} = \frac{\ln(3)}{2} \Rightarrow c - 1 = \frac{2}{\ln(3)}$

$\boxed{c = \frac{2}{\ln(3)} + 1}$

(8) $g(x) = \begin{cases} x^3 & -2 \le x \le 0 \\ x^2 & 0 < x \le 2 \end{cases}$

$g(2) = 2^2 = 4$ $\quad g(-2) = (-2)^3 = -8$ $\quad \frac{\Delta y}{\Delta x} = \frac{4-(-8)}{2-(-2)} = \frac{12}{4} = 3$

$g'(x) = \begin{cases} 3x^2 & -2 < x < 0 \\ 2x & 0 < x < 2 \end{cases}$

Note: if $c \in (-2, 0)$ then $g'(c) = 3c^2 = 3 \Rightarrow c = -1$
if $c \in (0, 2)$ then $g'(c) = 2c = 3 \Rightarrow c = 3/2$.

ie. there are 2 possible answers here.

(10) $f(x) = x^{4/5}$, $[0,1]$

This is a power function, so it is cts on $[0,1]$.
$f'(x) = \frac{4}{5} x^{-1/5}$, so it is diff'ble on $(0,1)$.
$\therefore$ f satisfies hypotheses for MVT.

(12) $f(x) = \begin{cases} \frac{\sin x}{x} & -\pi \le x < 0 \\ 0 & x = 0 \end{cases}$

$\lim_{x \to 0^-} f(x) = 1$ $f(0) = 0$ $\therefore$ f is not cts.

$\therefore$ f does not satisfy ~~hypotheses~~ for MVT.

(14) $f(x) = \begin{cases} 2x-3 & 0 \le x \le 2 \\ 6x - x^2 - 7 & 2 < x \le 3 \end{cases}$

only need to check at $x=2$ since f is polynomial everywhere else.

$f(2) = 2(2) - 3 = 1$

$\lim_{x \to 2^+} f(x) = 6(2) - 2^2 - 7 = 12 - 4 - 7 = 1$

f is cts on $[0,3]$

slope of tangent as $x \to 2^-$ is 2
slope of tangent as $x \to 2^+$ is $6 - 4x \Big|_{x=2} = 6 - 8 = -2$

f is not diff'ble at $x=2$.

$\therefore$ f does not satisfy hypotheses of MVT.

(16) $f(x) = \begin{cases} 3 & x = 0 \\ -x^2+3x+a & 0 < x < 1 \\ mx+b & 1 \le x \le 2 \end{cases}$

we want f to be cts on $[0,2]$ and diff'ble on $(0,2)$

$f(0) = 3$ $\quad \lim_{x\to 0^+} f(x) = a \quad \Rightarrow \quad \boxed{a=3}$

$f(1) = m+b$

$\lim_{x\to 1^-} f(x) = -1+3+a = -1+3+3 = 5 \quad \Rightarrow \quad m+b=5$

Slope of tangent line as $x \to 1^+$ is m

Slope of tangent line as $x \to 1^-$ is $-2x+3\Big|_{x=1} = -2+3 = 1$

$\therefore \boxed{m=1}$ $\quad \boxed{b=4}$

(22) $f(x) = x^3 + \frac{4}{x^2} + 7$, $(-\infty, 0)$

Show exactly one root.

First show there is a root.

f is cts on $(-\infty,0)$ since it is a rational function and root of numerator is not in domain.

$f(-1) = (-1)^3 + \frac{4}{(-1)^2} + 7 = -1+4+7 > 0$

$f(-3) = (-3)^3 + \frac{4}{(-3)^2} + 7 = -27 + \frac{4}{9} + 7 < 0$

Therefore f has a root c between -1 and -3 by the Intermediate Value Theorem.

To see that there are no more, proceed by proof by contradiction.

Suppose there is another root d.

Then by Rolle's Theorem (applicable since f is diff'ble)

there is a point e between c and d such that $f'(e)=0$.

We compute $f'(x) = 3x^2 - 4x^{-3} = \frac{3x^5-4}{x^3}$. This is 0 when $x = \sqrt[5]{\frac{4}{3}}$, which is not in domain.

Contradiction.

$\therefore$ f has exactly one real root.

(26) $r(\theta) = 2\theta - \cos^2\theta + \sqrt{2}$, $(-\infty, \infty)$

$r(0) = 0 - 1 + \sqrt{2} > 0$

$r(-100) = -200 - \cos^2(-100) + \sqrt{2} < 0$

By IVT (applicable since r is cts) we have a root c between 0 and -100.

To show there are no more, we use proof by contradiction. Suppose there is another root d.

The by Rolle's Theorem (applicable since r is diff'ble), we have a pt e between c and d such that $r'(c) = 0$.

We compute $r'(\theta) = 2 + 2\cos\theta\sin\theta. = 2 + \sin(2\theta)$

Note that $-1 \le \sin(2\theta) \le 1$, so $r'(\theta) \ge 1$. Contradiction.

Therefore r has exactly one real root.

(34) a) $y' = 2x$

$y = x^2 + C$, $C \in \mathbb{R}$

b) $y' = 2x - 1$

$y = x^2 - x + C$, $C \in \mathbb{R}$

c) $y' = 3x^2 + 2x - 1$

$y = x^3 + x^2 - x + C$, $C \in \mathbb{R}$

(40) $g'(x) = x^{-2} + 2x$ $\qquad P(-1, 1)$

$g(x) = -x^{-1} + x^2 + C$

$g(-1) = -(-1)^{-1} + (-1)^2 + C = 1 + 1 + C = 1 \Rightarrow C = -1$

$\boxed{g(x) = -x^{-1} + x^2 - 1}$

(44) $v = 32t - 2$ $\quad s(0.5) = 4$

$s(t) = 16t^2 - 2t + c$

$s(0.5) = 16(0.5)^2 - 2(0.5) + c = 4 - 1 + c = 4 \Rightarrow c = 1$

$\boxed{s(t) = 16t^2 - 2t + 1}$

(48) $a = 9.8$, $v(0) = -3$, $s(0) = 0$

$v(t) = 9.8t + c$

$v(0) = -3 \Rightarrow c = -3$

$v(t) = 9.8t - 3$

$s(t) = 4.9t^2 - 3t + c$

$s(0) = c = 0 \Rightarrow \boxed{s(t) = 4.9t^2 t^2 - 3t}$

(52) 2 hrs 159 miles.

$\text{aroc} = \frac{159}{2} = 79.5 \frac{\text{miles}}{\text{hr}}$

By MVT, there exist a time when the driver was travelling at $79.5 \frac{\text{miles}}{\text{hr}}$.

(54) 26.2 mi 2.2 hrs

$\text{aroc} = \frac{26.2}{2.2} = 11.9\ldots$ mi/hr. By MVT, there is a time t_0 the runner is going 11.9... mi/hr.

Speed is continuous. Speed at start and end is 0mph.

By IVT, there is a time between 0 and t_0 when she is going 11 mi/hr.

By IVT, there is a time between t_0 and end when she is going 4 mi/hr.