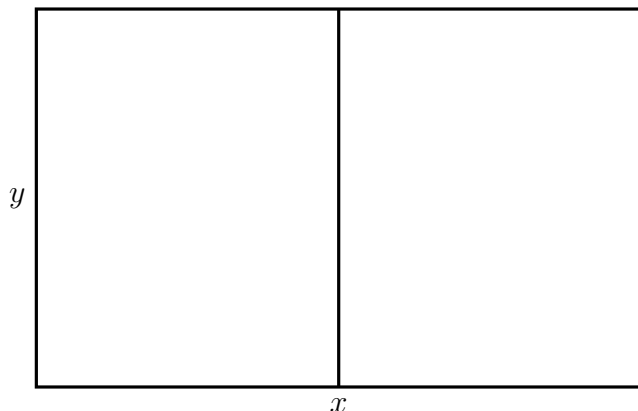# 4.6 SOLUTIONS/HINTS

DAN YASAKI

2. See lecture notes from 11/13/13.
4. See lecture notes from 11/13/13.
8. We draw a picture and define some variables. Let y and x be the lengths shown below.

Then the area A is given by $A = xy = 216$ m². We want to minimize the amount of fencing required. The length of required fencing F is $F = 2x + 3y$. Using the area constraint, we see that $y = 216/x$. We plug this in to F to get

$$F = 2x + 648x^{-1}.$$

From $xy = 216$, we see that $0 < x < \infty$. Thus, we want to minimize $F = 2x + 648x^{-1}$ on $0 < x < \infty$. We compute

$$\frac{dF}{dx} = 2 - 648x^{-2}.$$

Thus we have critical points when $x^2 = 324$. Thus $x = 18$ is the only critical point. Making a sign chart, we see that F is increasing on $(18, \infty)$ and decreasing on $(0, 18)$. Therefore we have a global minimum when $x = 18$.

Now we have to be sure that we answer the questions asked. Notice that since $xy = 216$, when $x = 18$, we have $y = 12$.

- The dimensions of the outer rectangle is 18 m × 12 m. (Be sure to include the units!)
- The fencing required is $2(18) + 3(12) = 72$ m.

12. For this problem, use the picture on page 269. The volume V of a cone of radius r and height h is $V = \frac{1}{3}\pi r^2 h$. We want to find the largest volume cone we can inscribe in a sphere of radius 3. Define x and y as shown in the picture. Then the height is $3 + y$ and the radius is x, so the volume is $V = \frac{1}{3}\pi x^2(3 + y)$.

Now we need to to try to understand the constraints. Notice that if x and y were not constrained, we could make the volume arbitrarily large. What is preventing

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that? It is the fact that the cone is inscribed in the sphere. Looking back at the picture, we see a nice right triangle that relates x, y and 3 via Pythagorean theorem. We have $x^2 + y^2 = 3^2 = 9$. In particular, $x^2 = 9 - y^2$ so the volume is given by

$$V = \frac{\pi}{3}(9 - y^2)(3 + y).$$

What are the constraints on y? Since $x^2 + y^2 = 9$, we see that $0 \le y \le 3$. Note that V is a continuous function on a closed interval, so we just need to find the critical points and plug in critical points and endpoints.

To make our life easier, we multiply out to find $V = \frac{\pi}{3}(27 + 9y - 3y^2 - y^3)$. Then we compute

$$\frac{dV}{dy} = \frac{\pi}{3}(9 - 6y - 3y^2) = -\pi(y^2 + 2y - 3) = -\pi(y + 3)(y - 1).$$

We have that $y = 1$ is the only critical point in the domain. We compute $V(0) = 9\pi$, $V(3) = 0$, and $V(1) = \frac{32\pi}{3}$ m^3. The largest volume cone is $\frac{32\pi}{3}$ m^3.

22. First we draw a picture and define some variables.
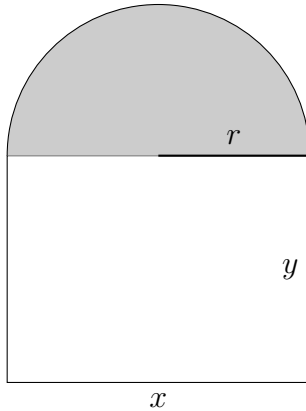
Note that $x = 2r$. Since the perimeter is fixed, we may as well fix it to be 1. Then we have

$$x + 2y + \pi r = (2 + \pi)r + 2y = 1.$$

Solving for y we get $y = (1 - (2 + \pi)r)/2$. We want to maximize the light. The tinted area is $\frac{\pi}{2}r^2$ and the rectangular area is $xy = 2ry$. Since the tinted glass lets through half the light, the light that gets through is

$$L = \frac{\pi}{4}r^2 + 2ry = \frac{\pi}{4}r^2 + r(1 - (2 + \pi)r) = r - (2 + \frac{3\pi}{4})r^2.$$

Since $(2 + \pi)r + 2y = 1$, we see that $0 \le r \le \frac{1}{2+\pi}$. We are minimizing a continuous function on a closed interval. We compute

$$\frac{dL}{dr} = 1 - 2(2 + \frac{3\pi}{4})r.$$

The critical point is $r = \frac{1}{4 + \frac{3\pi}{2}} = \frac{2}{8 + 3\pi}$ We could just plug in this critical point and the two endpoints, but it may be easier to make a sign chart for the derivative and note that L is increasing to the left of the critical point and decreasing to the right. It follows that we have a global maximum at $r = \frac{2}{8+3\pi}$. We now turn to answering

the question, which asks for the proportions of the window. Since $(2+\pi)r + 2y = 1$, when $r = \frac{2}{8+3\pi}$, we have

$$y = \frac{1 - (2+\pi)(\frac{2}{8+3\pi})}{2} = \frac{\frac{4+\pi}{8+3\pi}}{2}.$$

It follows that

$$\frac{x}{y} = \frac{\frac{4}{8+3\pi}}{\frac{4+\pi}{8+3\pi}} = \frac{4}{\frac{4+\pi}{2}} = \frac{8}{4+\pi}.$$

(Not: Since this is a proportion, there are no units to include here.)

24. For this problem, use the picture from page 270. We want to maximize the trough's volume V. We need to express the volume in terms of the variables introduced in the picture. The volume is the area of the cross-section times the length of the trough. First let's figure out the area of the cross-section.
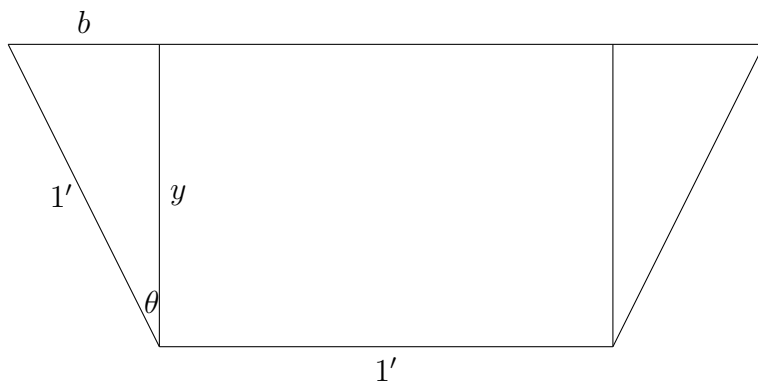
The area is $A = 1 \cdot y + by$ and so the volume is $V = 20(y + by)$. Now we need to get V as a function of one variable. Using trig (SOHCAHTOA), we see that $b = \sin\theta$ and $y = \cos\theta$. Thus

$$V = 20(\cos\theta + \sin\theta\cos\theta).$$

Since y and b are non-negative (or by looking at the picture), we see that $0 \le \theta \le \frac{\pi}{2}$. Since V is a continuous function on a closed interval, we just need to find critical points and check volume at these points along with the endpoints.
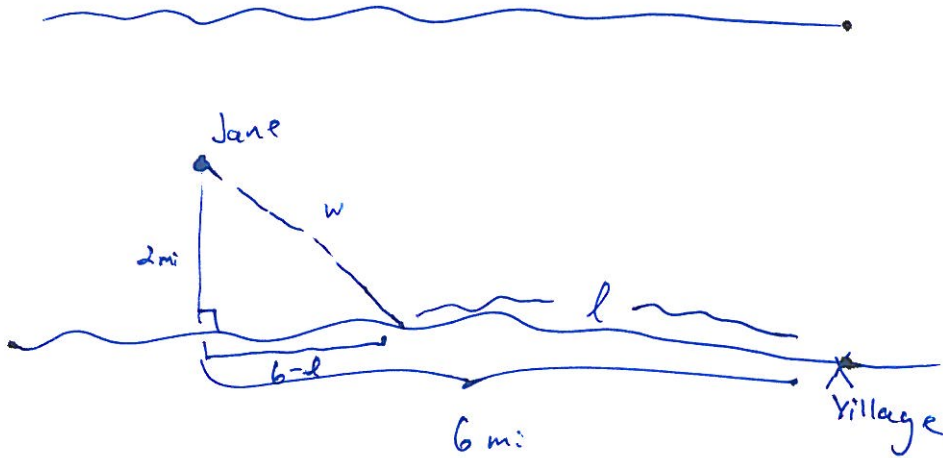
We compute

$$\frac{dV}{d\theta} = 20(-\sin\theta + \cos^2\theta - \sin^2\theta)$$
$$= -20(2\sin^2\theta + \sin\theta - 1)$$
$$= -20(\sin\theta + 1)(2\sin\theta - 1).$$

We have critical points where $\sin\theta = -1$ and where $\sin\theta = \frac{1}{2}$. The first equation has no solutions in the domain. The second equation is satisfied when $\theta = \frac{\pi}{6}$ We compute

$$V(0) = 20, \quad V(\frac{-2}{\pi}) = 0, \quad \text{and} \quad V(\frac{\pi}{6}) = 20(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4}).$$

Thus the volume is maximized when $\theta = \frac{\pi}{6}$.

Let w be the distance she rows, and let l be the distance she walks.

The time in the water is $t_w = \dfrac{w}{2}$ since she can row 2 mph.

Similarly the time on land is $t_l = \dfrac{l}{5}$ since she can walk 5 mph.

The total time is

$$t = \frac{w}{2} + \frac{l}{5} .$$

Now we need to find the constraints. Use Pyphagorean Theorem

$$(6-l)^2 + 2^2 = w^2 \quad \Rightarrow \quad w = \sqrt{(6-l)^2 + 4}$$

$$t = \frac{\sqrt{(6-l)^2+4}}{2} + \frac{l}{5} \qquad 0 \le l \le 6 \quad \text{from the picture.}$$

$$\frac{dt}{dl} = \frac{1}{2} \cdot \frac{1}{2}\left((6-l)^2+4\right)^{-1/2} \cdot \left(2(6-l)(-1)\right) + \frac{1}{5}$$

$$= \frac{(l-6)}{2\sqrt{(6-l)^2+4}} + \frac{1}{5}$$

$$= \frac{5(l-6) + 2\sqrt{(6-l)^2+4}}{10\sqrt{(6-l)^2+4}}$$

This is equal to 0 when

$$-5(6-\ell) + 2\sqrt{(6-\ell)^2+4} = 0$$

$$2\sqrt{(6-\ell)^2+4} = 5(6-\ell)$$

$$4[(6-\ell)^2+4] = 25(6-\ell)^2$$
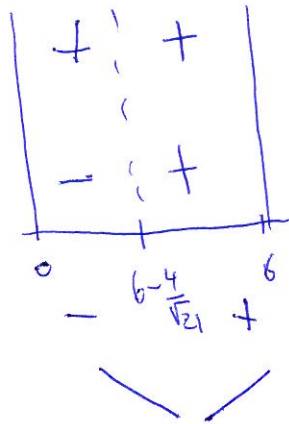
$$16 = 21(6-\ell)^2$$

$$\frac{16}{21} = (6-\ell)^2$$

$$\frac{4}{\sqrt{21}} = 6-\ell \qquad (\text{since} \quad \ell < 6)$$

$$\ell = 6 - \frac{4}{\sqrt{21}}$$

Using sign chart, we see t is decreasing to left and increasing to right of

$$\ell = 6 - \frac{4}{\sqrt{21}}$$

$$10\sqrt{(6-\ell)^2+4}$$

$$5(\ell-6) + 2\sqrt{(6-\ell)^2+4}$$

Thus we have a global minimum when $\ell = 6 - \frac{4}{\sqrt{21}}$.

In particular, she should land her boat $\frac{4}{\sqrt{21}}$ miles downshore.
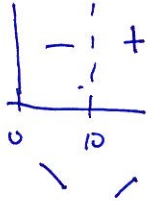
(54) The _average cost_ is

$$\bar{C}(x) = \frac{C(x)}{x} = \frac{x^3 - 20x^2 + 20000\,x}{x} = x^2 - 20x + 20000$$

$$0 < x < \infty$$

$$\bar{C}'(x) = 2x - 20$$

Critical pt $x = 10$

$2x - 20$

$\bar{C}$ has global minimum when
$x = 10$ items.

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