MAT 310 Test 1 February 17, 2014 (noon–12:50 PM)

Name: ________________________ Academic Integrity Signature: ________________________

I have abided by the UNCG Academic Integrity Policy.

Read all of the following information before starting the exam:

- It is to your advantage to answer ALL of the 19 questions.
- It is your responsibility to make sure that you have all of the problems.
- There is no need to complete the test in order. The problems are independent.
- Correct numerical answers with insufficient justification may receive little or no credit.
- Clearly distinguish your final answer from your scratch work with a box or circle.
- *Budget your time!*
- If you have read all of these instructions, draw a happy face here.

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Points:	16	15	11	8	15	15	14	6	100
Score:									

Definitions

1. (4 points) Let $S = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ be a nonempty set of vectors. The *span of* S, denoted Span(S), is

2. (4 points) A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ in $\mathbb{R}^n$ is said to be *linearly independent* if

3. (4 points) A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *linear* if

4. (4 points) (Be sure to give the definition and not one of the many equivalent statements.) An $n \times n$ matrix A is *invertible* or *nonsingular* if

 Points earned: ______ out of 16.

Examples and computations

5. (5 points) Give an example of an augmented matrix of an *inconsistent* linear system with 2 equations and 3 unknowns.

6. (5 points) Compute $T\left(\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\right)$, where $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation such that

$$T\left(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\right) = \begin{bmatrix}1\\2\\3\end{bmatrix} \quad \text{and} \quad T\left(\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\right) = \begin{bmatrix}2\\-9\\0\end{bmatrix}.$$

7. (5 points) Compute the inverse of $A = \begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\\1 & 1 & 1\end{bmatrix}$.

8. (6 points) Suppose A, B, and C are $n \times n$ matrices, with B invertible and

$$I - BAB^{-1} = C.$$

Solve this equation for A, or explain why there is no solution.

9. (5 points) Solve the linear system below. Give your answer in parametric form.

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 0 \\ 3x_1 + 5x_2 \phantom{{}- 2x_3} &= 1 \end{aligned}$$

 Points earned: ______ out of 11.

10. (8 points) Determine if each of the following sets are linearly independent or linearly dependent. Give reasons for your answers.

(a) $\left\{ \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ -4 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -3 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 9 \\ -4 \\ 4 \end{bmatrix} \right\}$

 Points earned: ______ out of 8.

11. (9 points) Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix}$, let $B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 3 & 2 \end{bmatrix}$, and let $\vec{v} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$. Compute the following, or explain why it is undefined.

(a) AB

(b) $A + B$

(c) $A\vec{v}$

12. (6 points) Give an example of a matrix A such that the linear transformation $T(\vec{x}) = A\vec{x}$ is surjective (onto), but NOT injective (one-to-one). Briefly explain how you know, for your particular choice of A, that (i) T is surjective, and (ii) T is not injective.

 Points earned: ______ out of 15.

Statements of Theorems and Results

13. (11 points) Answer each question by circling True if it *must* be true and False if it is *ever* false. No justification is required. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A.

    1. True | False: If T is injective, then A has a pivot in every column.
    2. True | False: If A has a pivot in every column, then T is surjective.
    3. True | False: If A has a pivot in every column, then $A\vec{x} = \vec{b}$ has exactly one solution for each $\vec{b} \in \mathbb{R}^m$.
    4. True | False: If A has a pivot in every row, then $A\vec{x} = \vec{b}$ is consistent for each $\vec{b} \in \mathbb{R}^m$.
    5. True | False: If the columns of A are linearly independent, then T is injective.
    6. True | False: Every set of 17 vectors in $\mathbb{R}^{31}$ is linearly independent.
    7. True | False: A linear system must have zero, one, or infinitely many solutions.
    8. True | False: Every homogeneous linear system is consistent.
    9. True | False: If a linear system has more equations than variables, then it is inconsistent.
    10. True | False: If two matrices have the same number of rows, then they are row-equivalent.
    11. True | False: T is injective if and only of $T(\vec{x}) = \vec{0}$ has only the trivial solution.

14. (4 points) Let A and B be invertible $n \times n$ matrices. For each of the following, circle the statement that must be true. No justification is required.

    (a) 1. $A + B$ is invertible, and $(A+B)^{-1} = A^{-1} + B^{-1}$.
    2. $A + B$ is invertible, and $(A+B)^{-1} = -A - B$.
    3. $A + B$ is not necessarily invertible.

    (b) 1. AB is invertible, and $(AB)^{-1} = A^{-1}B^{-1}$.
    2. AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.
    3. AB is not necessarily invertible.

    (c) 1. A^T is invertible, and $(A^T)^{-1} = (A^{-1})^T$.
    2. A^T is invertible, and $(A^T)^{-1} = -A^T$.
    3. A^T is not necessarily invertible.

    (d) 1. $(AB)^T = A^T B^T$.
    2. $(AB)^T = B^T A^T$.
    3. $(AB)^T = (AB)^{-1}$.

 Points earned: ______ out of 15.

Proofs

15. Let A be an $m \times n$ matrix with linearly independent columns $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$. Let T be the linear map defined by $T(\vec{x}) = A\vec{x}$.

(a) (2 points) What is the domain and codomain of T?

domain = ☐ codomain = ☐

(b) (5 points) Prove that T is injective (one-to-one).

16. (7 points) Suppose that $S\colon \mathbb{R}^n \to \mathbb{R}^p$ and $T\colon \mathbb{R}^p \to \mathbb{R}^m$ are linear maps. Let P be the composition $P(\vec{x}) = T(S(\vec{x}))$. Prove that P is a linear transformation.

 Points earned: ______ out of 14.

17. (6 points) Let $A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \\ -1 & -3 \end{bmatrix}$ and let $\vec{b} = \begin{bmatrix} 2 \\ -7 \\ 4 \end{bmatrix}$.

   (a) Is $\vec{b}$ in the span of the columns of A? Justify.

   (b) Define the linear map $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\vec{x}) = A\vec{x}$. Is T surjective (onto)? Justify.

Bonus

18. (1 point (bonus)) Tell me a funny joke.

19. (1 point (bonus)) In order to solve the equation $x^2 = -1$, mathematicians needed to invent a number i which is not in $\mathbb{R}$. Show that we can avoid this issue if we replace $\mathbb{R}$ by 2×2 matrices. Specifically, find a 2×2 matrix with real entries that satisfies $X^2 = -I$.

 Points earned: ______ out of 6.