PROOF BY INDUCTION

Read Chapter 2 (especially 2.3) of *The Art of Proof* by Matthias Beck and Ross Geoghegan

`http://link.springer.com/book/10.1007%2F978-1-4419-7023-7`

You can also access this book while off-campus by searching the UNCG library catalog online for that title.

Principle of mathematical induction. Let $P(k)$ be a statement, depending on a variable $k \in \mathbb{Z}$, that makes sense for all $k \geq m$, where m is a fixed integer. In order to prove the statement "$P(k)$ is true for all $k \geq m$" it is sufficient to prove

(1) $P(m)$ is true. (base case)
(2) For any $n \geq m$, if $P(n)$ is true, then $P(n+1)$ is true. (induction step)

Template for proof by induction.

(1) Formulate $P(k)$.
(2) Base case: Prove $P(m)$ is true, where m is the fixed integer, usually 0 or 1.
(3) Induction step: Let $n \in \mathbb{Z}$, $n > m$. Assuming $P(k)$ is true for all $m \leq k < n$, prove that $P(n)$ is true.

Theorem 1. *If A is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then*

$$\det(EA) = \det(E)\det(A),$$

where

$$\det(E) = \begin{cases} 1 & \textit{if } E \textit{ is row replacement,} \\ -1 & \textit{if } E \textit{ is interchange,} \\ r & \textit{if } E \textit{ is scale by } r. \end{cases}$$

Proof. The proof is by induction on the size of A. Let $P(k)$ be the statement of the theorem for $k \times k$ matrices, for $k \geq 1$. Let $B = EA$.

(Base case) In case $k = 1$, there is no row replacement and no interchange. The result is clear for scaling in this case. To get the induction going, we need to look at the next case as the base case. We must prove the statement for 2×2 matrices. This was exercise 33–36 of 3.1. [Here, $m = 2$ in the template. Let me know if you want to see details.]

(Induction step) Fix $n > 2$. Suppose $P(k)$ is true for all $2 \leq k < n$. We want to prove $P(n)$ is true. The action of E on A involves either two rows (interchange) of one row (scale or row replacement). Since $n > 2$, there is a row that is unchanged by the action of E. Call it row i. Let A_{ij} (respectively, B_{ij}) be the matrix obtained by deleting row i and column j from A (respectively B). Then B_{ij} is an $(n-1) \times (n-1)$ matrix that is obtained from A_{ij} by the same type of elementary row operation. The induction hypothesis implies

$$\det(B_{ij}) = \alpha \det A_{ij},$$

where $\alpha = 1$, -1, or r depending on the type of E. The cofactor expansion along row i is

$$\begin{aligned}\det(EA) &= \sum_{j=1}^{n} a_{ij}(-1)^{i+j}\det(B_{ij}) \\ &= \sum_{j=1}^{n} a_{ij}(-1)^{i+j}\alpha\det(A_{ij}) \\ &= \alpha\sum_{j=1}^{n} a_{ij}(-1)^{i+j}\det(A_{ij}) \\ &= \alpha\det(A),\end{aligned}$$

and so $P(n)$ is true. Furthermore, using $A = I$, we see that $\det(E) = \alpha$. By the principle of mathematical induction, $P(n)$ must be true for $n \geq 2$. The theorem is trivially true for $n = 1$, which finishes the proof. □

Theorem 2. *If A and B are square matrices, then* $\det(AB) = \det(A)\det(B)$.

Proof. Let $P(k)$ be the statement "If A can be expressed as a product of k elementary matrices $A = E_k E_{k-1} \ldots E_1$, then $\det(AB) = \det(A)$."

We will prove $P(k)$ for all $k \geq 1$ by induction.

(Base case) $P(1)$ is just Theorem 1.

(Induction step) Complete this proof at home. Come see me or go to the Math Help Center if you get stuck.

□