Differential Forms

MA 305
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Aesthetic pleasure needs no justification, because a life without such pleasure is one not worth living. Dana Gioia, "Can Poetry Matter?"

Introduction

We've hit the big three theorems of vector calculus: Green's, Stokes', and the Divergence Theorem (although Green's theorem is really a pretty obvious special case of Stokes' theorem). But it's an amazing fact that all of these theorems are really special cases of an even larger theorem that unifies all that we've studied so far. This new theory also extends what we've done to spaces of arbitrary (finite) dimension. It rests on the idea of *differential forms*.

Differential forms are hard to motivate right off the bat. It won't be immediately clear how differential forms are related to what we've done, but be patient. For now treat differential forms as a new kind of mathematical object; we're going to learn how to manipulate them—add, multiply, differentiate, and integrate. It will be quite abstract and "formal" at first, but you'll soon see how they connect with what we've already done.

Manifolds

You've already seen many examples of one and two dimensional manifolds. A one dimensional manifold C is just a curve. Such a curve can exist in any dimension and can be parameterized as $\mathbf{X}(t)$ (I'm using this instead of $\mathbf{R}$), where t ranges over some interval $D = (a, b)$ in $\mathbb{R}^1$ and

$$\mathbf{X}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$$

if the curve exists in n dimensions. You may recall that we required that our curves be smooth, which meant that the curve had to have a proper parameterization—$\mathbf{X}(t)$ must be differentiable, injective, and $\mathbf{X}'(t) \neq 0$ for any t in D. Smooth one dimensional manifolds are just smooth curves. Recall that the vector $\mathbf{X}'(t)$ is tangent to the manifold C.

A smooth two dimensional manifold S is just a surface, of which we've seen many examples (in three dimensions.) Such a surface can be specified as $\mathbf{X}(u, v)$, where (u, v) ranges over some appropriate subset D of $\mathbb{R}^2$. The function $\mathbf{X}(u, v)$ is of the form

$$\mathbf{X}(u, v) = (x_1(u, v), x_2(u, v), \ldots, x_n(u, v)),$$

although we've mostly considered the case $n = 3$. There were some technicalities in dealing with surfaces and integrating over them—the surface had to be orientable, and we did require that the tangent vectors $\frac{\partial \mathbf{X}}{\partial u}$ and $\frac{\partial \mathbf{X}}{\partial v}$ be "independent", i.e., neither a multiple of the other. If this isn't true then our surface S might end up being only one-dimensional. Also, the surface was not allowed to self-intersect.

Based on the above, it's not much of a stretch to consider a k-dimensional manifold. Let D be some bounded open region in $\mathbb{R}^k$. We'll use coordinates $\mathbf{u} = (u_1, u_2, \ldots, u_k)$ on D. Consider a function

$$\mathbf{X}(\mathbf{u}) = (X_1(\mathbf{u}), \ldots, X_n(\mathbf{u}))$$

defined on D, which maps each point in D into a point in n dimensional space. Take $M = \mathbf{X}(D)$, the image of D in $\mathbb{R}^n$. We will say that M is a smooth k-manifold if

- The function $\mathbf{X}$ is C^1.
- The function $\mathbf{X}$ is injective (distinct points in D go to distinct points in $\mathbb{R}^n$.
- The tangent vectors
$$\frac{\partial \mathbf{X}}{\partial u_k} = \left(\frac{\partial X_1}{\partial u_k}, \ldots, \frac{\partial X_n}{\partial u_k} \right)$$
are all linearly independent at all points in D. This means that the only solution to
$$c_1 \frac{\partial X_1}{\partial u_k} + \cdots + c_n \frac{\partial X_n}{\partial u_k} = 0$$
is $c_1 = c_2 = \cdots = c_n = 0$. Since this is a semi-intuitive account of differential forms and manifolds, we won't harp too much on this last requirement.

Example 1: Let D be the open cube $0 < u_1, u_2, u_3 < 1$ in $\mathbb{R}^3$. Take the mapping $\mathbf{X}$ from $\mathbb{R}^3$ to $\mathbb{R}^5$ to be defined by

$$\begin{aligned} X_1(u_1, u_2, u_3) &= u_1 u_2 \\ X_2(u_1, u_2, u_3) &= 3u_3 \\ X_3(u_1, u_2, u_3) &= u_1^2 \\ X_4(u_1, u_2, u_3) &= u_3 - 2u_2 \\ X_5(u_1, u_2, u_3) &= 3. \end{aligned}$$

The function $\mathbf{X}$ is obviously C^1. It's a bit tedious (Maple makes it easier) but you can check that no two points in D go to the same image point in $\mathbb{R}^5$, and the tangent vectors are in fact all independent. The image $M = \mathbf{X}(D)$ is a smooth 3-manifold in $\mathbb{R}^5$.

Differential Forms

Our discussions will take place in $\mathbb{R}^n$. We'll use $(x_1, \ldots, x_n)$ as coordinates, and write $\mathbf{x}$ for the point $(x_1, \ldots, x_n)$. We will define a 0-form ω to be simply a function on $\mathbb{R}^n$, so

$$\omega = f(x_1, \ldots, x_n).$$

That's all there is to say about 0-forms.

A *basic* or *elementary* 1-form in $\mathbb{R}^n$ is an expression like dx_i, where $1 \le i \le n$. More generally, a 1-form ω is an expression like

$$\omega = F_1(\mathbf{x})\, dx_1 + F_2(\mathbf{x})\, dx_2 + \cdots F_n(\mathbf{x})\, dx_n$$

where the F_i are functions of $\mathbf{x}$. You've actually encountered 1-forms before, when we did Green's theorem. Recall that Green's Theorem said

$$\int_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy = \int_{\partial D} F_1(x, y)\, dx + F_2(x, y)\, dy.$$

The integrand on the right is an example of a 1-form.

A differential 1-form is not a passive object, but in fact can be thought of as a kind of "function." The basic 1-form dx_i accepts as input a single vector $\mathbf{v}$ and outputs v_i, the ith component of $\mathbf{v}$, so

$$dx_i(\mathbf{v}) = v_i.$$

A general 1-form $\omega = F_1(\mathbf{x})\, dx_1 + \cdots F_n(\mathbf{x})\, dx_n$ acts on a single input vector $\mathbf{v}$ as

$$\omega(\mathbf{v}) = F_1(\mathbf{x}) v_1 + \cdots F_n(\mathbf{x}) v_n.$$

You can add two 0-forms in the obvious way (as functions). You can similarly add two 1-forms, e.g.,

$$(x_2^2\, dx_1 + e^{x_1}\, dx_2) + (2\, dx_1 - x_1\, dx_2) = (x_2^2 + 2)\, dx_1 + (e^{x_1} - x_1)\, dx_2.$$

A basic differential 2-form ω is an expression like

$$\omega = dx_i \wedge dx_j$$

where $1 \leq i, j \leq n$. The symbol $\wedge$ denotes what is called the *wedge* or *exterior* product. Don't worry too much about what it means (yet); in the end it will mean essentially just $dx_i dx_j$, an object suitable to stick under a double integral.

Like 1-forms, 2-forms also act on vectors. A basic two form $\omega = dx_i \wedge dx_j$ accepts as input TWO vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. The output is the determinant

$$\omega(\mathbf{v}_1, \mathbf{v}_2) = dx_i \wedge dx_j(\mathbf{v}_1, \mathbf{v}_2) = \det \begin{bmatrix} dx_i(\mathbf{v}_1) & dx_i(\mathbf{v}_2) \\ dx_j(\mathbf{v}_1) & dx_j(\mathbf{v}_2) \end{bmatrix}$$

Based on the properties for the determinant that we've seen, you can immediately conclude that

- $dx_i \wedge dx_j = -dx_j \wedge dx_i$
- $dx_i \wedge dx_i = 0$

This raises an interesting question: "How many different 2-forms are there in n dimensions?" If you count all possible combinations for $1 \leq i, j \leq n$ for $dx_i \wedge dx_j$ you get n^2, but in fact n of these are really zero. That leaves $n^2 - n$ possible 2-forms, although in some sense there is only half that many, for the second property above shows that, e.g., $dx_1 \wedge dx_2 = -dx_2 \wedge dx_1$. There are thus only $(n^2 - n)/2$ half this many "independent" 2-forms.

A (general) differential 2-form is an expression of the form

$$\omega = F_{12}(\mathbf{x}) dx_1 \wedge dx_2 + F_{13}(\mathbf{x}) dx_1 \wedge dx_3 + \cdots F_{n-1,n}(\mathbf{x}) dx_{n-1} \wedge dx_n = \sum_{1 \leq i < j \leq n} F_{ij}(\mathbf{x}) dx_i \wedge dx_j.$$

Notice that we can always assume that $i < j$ in the sum above, for if we had $j < i$ we could simply swap the order of $dx_j \wedge dx_i$ and replace it with $-dx_i \wedge dx_j$.

You can probably guess how such a 2-form acts on two input vectors:

$$\omega(\mathbf{v}_1, \mathbf{v}_2) = \sum_{1 \le i < j \le n} F_{ij}(\mathbf{x}) dx_i \wedge dx_j(\mathbf{v}_1, \mathbf{v}_2) = \sum_{i, j > i} F_{ij}(\mathbf{x}) \det \begin{bmatrix} dx_i(\mathbf{v}_1) & dx_i(\mathbf{v}_2) \\ dx_j(\mathbf{v}_1) & dx_j(\mathbf{v}_2) \end{bmatrix}$$

Example 2: Let ω denote the 2-form in three dimensions

$$\omega = (x_1 + 2x_3) dx_1 \wedge dx_2 + x_2 dx_1 \wedge dx_3.$$

Let's compute ω applied to the vectors $\mathbf{v}_1 = (1, 3, 3)$, $\mathbf{v}_2 = (-1, 0, 7)$. You get

$$\begin{aligned} \omega(\mathbf{v}_1, \mathbf{v}_2) &= (x_1 + 2x_3) \det \begin{bmatrix} dx_1(\mathbf{v}_1) & dx_1(\mathbf{v}_2) \\ dx_2(\mathbf{v}_1) & dx_2(\mathbf{v}_2) \end{bmatrix} + x_2 \det \begin{bmatrix} dx_1(\mathbf{v}_1) & dx_1(\mathbf{v}_2) \\ dx_3(\mathbf{v}_1) & dx_3(\mathbf{v}_2) \end{bmatrix} \\ &= (x_1 + 2x_3) \det \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix} + x_2 \det \begin{bmatrix} 1 & -1 \\ 3 & 7 \end{bmatrix} \\ &= 3(x_1 + 2x_3) + 10x_2 = 3x_1 + 10x_2 + 6x_3. \end{aligned}$$

Now we're ready for the definition of a basic k-form: a basic k-form ω in n dimensions is an expression of the form

$$\omega = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$$

where $1 \le i_j \le n$ for all j. Such a k-form accepts as input k vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ to give output

$$\omega(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \det \begin{bmatrix} dx_{i_1}(\mathbf{v}_1) & dx_{i_1}(\mathbf{v}_2) & \cdots & dx_{i_1}(\mathbf{v}_k) \\ dx_{i_2}(\mathbf{v}_1) & dx_{i_2}(\mathbf{v}_2) & \cdots & dx_{i_2}(\mathbf{v}_k) \\ \vdots & \vdots & \vdots & \vdots \\ dx_{i_k}(\mathbf{v}_1) & dx_{i_k}(\mathbf{v}_2) & \cdots & dx_{i_k}(\mathbf{v}_k) \end{bmatrix}$$

By using the properties of determinants that we've already deduced, it's easy to see that

$$\begin{aligned} dx_{i_1} \wedge \cdots \wedge dx_{i_j} \wedge \cdots dx_{i_k} \wedge \cdots \wedge dx_{i_m} &= -dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge \cdots dx_{i_j} \wedge \cdots \wedge dx_{i_m} \quad (1) \\ dx_{i_1} \wedge \cdots \wedge dx_{i_j} \wedge \cdots dx_{i_j} \wedge \cdots \wedge dx_{i_m} &= 0 \quad (2) \end{aligned}$$

Given the above two facts, it's interesting to contemplate the question "how many independent basic k-forms are there in n dimensions?" It's not too hard to figure out, so I'll leave it for you.

A (general) k-form ω is an expression of the form

$$\omega = \sum_{1 \le i_1, \ldots, i_k \le n} F_{i_1, \ldots, i_k}(\mathbf{x}) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}. \quad (3)$$

Such a k-form accepts k input vectors to produce

$$\omega(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \sum_{1 \le i_1, \ldots, i_k \le n} F_{i_1, \ldots, i_k}(\mathbf{x}) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}(\mathbf{v}_1, \ldots, \mathbf{v}_k).$$

With this notation, it's easy to see why a function is called a 0-form—it doesn't accept any input vectors.

Integrating Differential Forms

Integrating differential forms is easy. The general rule is that one integrates a k-form over a k-manifold.

Let's first look at how to integrate a 1-form over a 1-manifold. Accordingly, let $M = \mathbf{X}(t)$ with $a \le t \le b$ be a smooth 1-manifold in $\mathbb{R}^n$ and let ω be a 1-form defined in a neighborhood of M. We use $\int_M \omega$ to denote the integral of ω over M and define this integral by

$$\int_M \omega = \int_a^b \omega(\mathbf{X}'(t))\,dt. \tag{4}$$

Example 3: Let M be the 1-manifold in $\mathbb{R}^3$ defined by $\mathbf{X}(t) = (3t, t^2, 5-t)$ for $0 \le t \le 2$. You can see that M is just a curve in three dimensions. Let ω be the 1-form $\omega = 2x_2dx_1 - x_1x_3dx_2 + dx_3$. Using equation (4) and recalling how a 1-form acts on input vectors we obtain

$$\begin{aligned}\int_M \omega = \int_0^2 \omega((3,2t,-1))\,dt &= \int_0^2 ((2x_2)(3) - x_1x_3(2t) - 1)\,dt \\ &= \int_0^2 (6t^3 - 24t^2 - 1)\,dt = -42.\end{aligned}$$

Example 4: Suppose that $\mathbf{F}(\mathbf{x}) = (F_1, F_2, \ldots, F_n)$ is a vector field in $\mathbb{R}^n$ (so each of the F_i are functions of $\mathbf{x}$). Define the differential form

$$\omega = F_1dx_1 + F_2dx_2 + \cdots + F_ndx_n.$$

If we compute the integral of ω over some 1-manifold defined by $M = \mathbf{R}(t) = (x_1(t), \ldots, x_n(t))$ we get

$$\begin{aligned}\int_M \omega &= \int_a^b (F_1dx_1 + F_2dx_2 + \cdots + F_ndx_n)(\mathbf{R}'(t))\,dt \\ &= \int_a^b (F_1\frac{\partial x_1}{\partial t} + \cdots + F_n\frac{\partial x_n}{\partial t})\,dt = \int_M \mathbf{F}\cdot d\mathbf{R}.\end{aligned}$$

In other words, the integral of the 1-form ω over M is just the familiar line integral of the vector field $\mathbf{F}$ over the curve M.

One can integrate a 2-form ω over a 2-manifold M. Suppose that M is parameterized by $M = X(\mathbf{u})$, where $\mathbf{u} = (u_1, u_2)$ ranges over D, a region in $\mathbb{R}^2$. The integral is defined by

$$\int_M \omega = \int_D \omega(\frac{\partial \mathbf{X}}{\partial u_1}, \frac{\partial \mathbf{X}}{\partial u_2})\,du_1\,du_2.$$

Example 5: Let M be a 2-manifold in $\mathbb{R}^4$ parameterized by

$$\mathbf{X}(\mathbf{u}) = (u_1, u_1 - u_2, 3 - u_1 + u_1u_2, -3u_2)$$

where $u_1^2+u_2^2<1$. Take $\omega = x_2\,dx_1\wedge dx_3 - x_4 dx_3\wedge dx_4$. You can easily compute that

$$\begin{aligned}\frac{\partial \mathbf{X}}{\partial u_1} &= (1,1,u_2-1,0)\\ \frac{\partial \mathbf{X}}{\partial u_2} &= (0,-1,u_1,-3)\end{aligned}$$

and that

$$\begin{aligned}\omega(\frac{\partial \mathbf{X}}{\partial u_1},\frac{\partial \mathbf{X}}{\partial u_2}) &= x_2\det\begin{bmatrix}1 & 0\\ u_2-1 & u_1\end{bmatrix} - x_4\det\begin{bmatrix}u_2-1 & u_1\\ 0 & -3\end{bmatrix}\\ &= x_2u_1+3x_4(u_2-1)\\ &= u_1^2-u_1u_2-9u_2^2+9u_2.\end{aligned}$$

As a result

$$\int_M \omega = \int_{-1}^{1}\int_{-\sqrt{1-u_1^2}}^{\sqrt{1-u_1^2}} (u_1^2-u_1u_2-9u_2^2+9u_2)\,du_2\,du_1 = -2\pi.$$

Example 6: Consider a smooth 2-manifold M in $\mathbb{R}^3$ parameterized as $\mathbf{X}(\mathbf{u})$ for $\mathbf{u}$ in $D\subset\mathbb{R}^2$. Let

$$\omega = F_1dx_2\wedge dx_3 + F_2dx_1\wedge dx_3 + F_3dx_1\wedge dx_2$$

be a 2-form defined in a neighborhood of M. Then

$$\begin{aligned}\int_M\omega &= \int_D F_1(dx_2\wedge dx_3)(\frac{\partial \mathbf{X}}{\partial u_1},\frac{\partial \mathbf{X}}{\partial u_2}) + F_2(dx_1\wedge dx_3)(\frac{\partial \mathbf{X}}{\partial u_1},\frac{\partial \mathbf{X}}{\partial u_2}) + F_3(dx_1\wedge dx_2)(\frac{\partial \mathbf{X}}{\partial u_1},\frac{\partial \mathbf{X}}{\partial u_2})\\ &= \int_D\left(F_1\det\begin{bmatrix}\frac{\partial X_2}{\partial u_1} & \frac{\partial X_2}{\partial u_2}\\ \frac{\partial X_3}{\partial u_1} & \frac{\partial X_3}{\partial u_2}\end{bmatrix} + F_2\det\begin{bmatrix}\frac{\partial X_1}{\partial u_1} & \frac{\partial X_1}{\partial u_2}\\ \frac{\partial X_3}{\partial u_1} & \frac{\partial X_3}{\partial u_2}\end{bmatrix} + F_3\det\begin{bmatrix}\frac{\partial X_1}{\partial u_1} & \frac{\partial X_1}{\partial u_2}\\ \frac{\partial X_2}{\partial u_1} & \frac{\partial X_2}{\partial u_2}\end{bmatrix}\right)du_1\,du_2\\ &= \int_D F_1(\frac{\partial X_2}{\partial u_1}\frac{\partial X_3}{\partial u_2} - \frac{\partial X_2}{\partial u_2}\frac{\partial X_3}{\partial u_1}) + F_2(\frac{\partial X_1}{\partial u_1}\frac{\partial X_3}{\partial u_2} - \frac{\partial X_1}{\partial u_2}\frac{\partial X_3}{\partial u_1})\\ &\quad +F_3(\frac{\partial X_1}{\partial u_1}\frac{\partial X_2}{\partial u_2} - \frac{\partial X_1}{\partial u_2}\frac{\partial X_2}{\partial u_1})\,du_1\,du_2\\ &= \int_M \mathbf{F}\cdot d\mathbf{A}.\end{aligned}$$

In other words, the integral of ω over M is just the flux of the vector field $\mathbf{F} = (F_1,F_2,F_3)$ over M. If the step between the last two equations looks mysterious, all I've done is use the fact that $d\mathbf{A} = (\partial\mathbf{X}/\partial u_1)\times(\partial\mathbf{X}/\partial u_2)$ and dotted this with $\mathbf{F}$; it's the usual procedure for computing flux over a parameterized surface.

You can probably see how we should integrate a general k-form over a k-manifold. If the manifold M is parameterized by $\mathbf{X}(\mathbf{u})$ for $u\in D\subset\mathbb{R}^k$, and if we have a general k-form ω given by an equation like (3) then

$$\int_M\omega = \sum_{1\le i_1,\ldots,i_k\le n}\int_D F_{i_1,\ldots,i_k}(\mathbf{X}(\mathbf{u}))(dx_{i_1}\wedge\cdots\wedge dx_{i_k})(\frac{\partial \mathbf{X}}{\partial u_1},\ldots,\frac{\partial \mathbf{X}}{\partial u_k})\,du_1\cdots du_k.$$

Example 7: Let M be a 3-manifold in $\mathbb{R}^4$ parameterized as

$$\mathbf{X}(u_1, u_2, u_3) = (u_2u_3, u_1^2, 1 - 3u_2 + u_3, u_1u_2)$$

for $u_1^2 + u_2^2 + u_3^2 < 1$. Take

$$\omega = x_3\, dx_1 \wedge dx_2 \wedge dx_4.$$

It's easy to compute that

$$\begin{aligned} \frac{\partial \mathbf{X}}{\partial u_1} &= (0, 2u_1, 0, u_2) \\ \frac{\partial \mathbf{X}}{\partial u_2} &= (u_3, 0, -3, u_1) \\ \frac{\partial \mathbf{X}}{\partial u_3} &= (u_2, 0, 1, 0). \end{aligned}$$

Then

$$\omega(\frac{\partial \mathbf{X}}{\partial u_1}, \frac{\partial \mathbf{X}}{\partial u_2}, \frac{\partial \mathbf{X}}{\partial u_3}) = x_3 \det \begin{bmatrix} 1 & u_3 & u_2 \\ 2u_1 & 0 & 0 \\ u_2 & u_1 & 0 \end{bmatrix} = 2u_1^2u_2 - 6u_1^2u_2^2 + 2u_1^2u_2u_3.$$

Finally, we find that

$$\int_M \omega = \int\int\int_D (2u_1^2u_2 - 6u_1^2u_2^2 + 2u_1^2u_2u_3)\, du_1\, du_2\, du_3 = -\frac{8}{35}\pi.$$

The last triple integral is easier if you convert it to spherical.

Example 8: Let M be an open region in $\mathbb{R}^n$ and suppose that M is parameterized as $\mathbf{X}(\mathbf{u})$ where $\mathbf{u} \in D$, where D is another region in $\mathbb{R}^n$. In other words, M is an n-manifold in n-dimensional space. Of course, we could very easily parameterize M by taking $D = M$ and using $\mathbf{X}(\mathbf{u}) = \mathbf{u}$, the identity map, but this isn't necessary, and by not doing so you'll see some important issues in integrating differential forms over manifolds.

Let $\omega = f(\mathbf{x})\, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ (this is the most general n-form in $\mathbb{R}^n$—why?) Then

$$\begin{aligned} \int_M \omega &= \int_D f(\mathbf{X}(\mathbf{u}))(dx_1 \wedge \cdots \wedge dx_n)(\frac{\partial \mathbf{X}}{\partial u_1}, \ldots, \frac{\partial \mathbf{X}}{\partial u_n}) \\ &= \int_D f(\mathbf{X}(\mathbf{u})) \det \begin{bmatrix} \frac{\partial X_1}{\partial u_1} & \cdots & \frac{\partial X_n}{\partial u_1} \\ \vdots & \vdots & \vdots \\ \frac{\partial X_1}{\partial u_n} & \cdots & \frac{\partial X_n}{\partial u_n} \end{bmatrix} du_1 \ldots du_n \\ &= \pm \int_M f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n \end{aligned}$$

where the last equality follows from the change of variables formula in n dimensions (notice that the absolute value signs on the determinant are missing). The integral of ω over M is PLUS OR MINUS the integral of f over M. It could be either, depending on the parameterization.

The last example raises an important issue that we won't entirely resolve. You might hope the quantity $\int_M \omega$ should depend only on M and ω, but this is true only up to a minus sign. The problem is that any manifold which is orientable has two possible orientations, and this is where the sign ambiguity arises. You've already encountered this in line and surface integrals: if C is a curve in two or three dimensions and $\mathbf{F}$ is a vector field, the quantity $\int_C \mathbf{F} \cdot d\mathbf{R}$ is not well-defined. It depends on which direction you parameterize or traverse the curve C; if you consider this integral physically, as work done, it clearly depends on whether you're moving from point A to point B or the reverse. The sign ambiguity also arose in doing flux integrals over a surface. Which direction you choose for the normal affects the sign of the answer.

It is a general fact (that I'll let you check for yourself) that the quantity $\int_M \omega$ does not depend on how M is parameterized, except for the sign of the answer. Proving this really comes down to the change of variable formula for integral of n variables. The different signs are a manifestation of how we chose to orient M (implicitly) when we parameterized it.

The Wedge Product

The wedge (or *exterior* product allows us to "multiply" differential forms. Suppose that f is a 0-form (a function) and ω is a k-form as given in equation (3). The wedge product of f with ω is just

$$f \wedge \omega = \sum_{i_1,\dots,i_k} f F_{i_1,\dots,i_k}(\mathbf{x}) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}.$$

Suppose that η is an m-form,

$$\eta = \sum_{j_1,\dots,j_m} G_{j_1,\dots,j_m}(\mathbf{x}) dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_m}.$$

The wedge product of η with ω is the $k+m$ form given by "adjoining" the two forms, as

$$\eta \wedge \omega = \sum F_{i_1,\dots,i_k}(\mathbf{x}) G_{j_1,\dots,j_m}(\mathbf{x}) dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_m}.$$

As complicated as this looks, it's pretty easy in any specific example, and the rules (1) and (2) make the result a lot smaller than you might think.

> **Example 9:** Let $\omega = x_2\, dx_1 \wedge dx_3 + dx_1 \wedge dx_4$ and $\eta = (x_1+1) dx_2 \wedge dx_4$. Then
>
> $$\omega \wedge \eta = x_2(x_1+1)\, dx_1 \wedge dx_3 \wedge dx_2 \wedge dx_4 + (x_1+1)\, dx_1 \wedge dx_4 \wedge dx_2 \wedge dx_4.$$
>
> However, by equation (2) the second term above (which contains two copies of dx_4) is zero. Also, by property (1) we can flip the dx_2 and dx_3 in the first term, which will introduce a minus sign, so
>
> $$\omega \wedge \eta = -x_2(x_1+1)\, dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4.$$

It's not hard to prove the following properties for the wedge product: If f is a function, ω_1 and ω_2 are k-forms, η an m-form, and τ any form then

$$(\omega_1 + \omega_2) \wedge \eta \quad = \quad \omega_1 \wedge \eta + \omega_2 \wedge \eta$$

$$\begin{aligned}
(\omega_1 \wedge \eta) \wedge \tau &= \omega_1 \wedge (\eta \wedge \tau) \\
(f\omega_1) \wedge \eta &= f(\omega_1 \wedge \eta) = \omega_1 \wedge (f\eta) \\
\omega_1 \wedge \eta &= (-1)^{km} \eta \wedge \omega_1
\end{aligned}$$

Only the last property is at all "nonobvious", but a little experimentation will show you how to prove it. In summary, this last property says that $\omega \wedge \eta = \eta \wedge \omega$ UNLESS both forms are of odd degree, in which case the sign flips.

The Exterior Derivative

Given that differential forms can be integrated, it stands to reason that they can also be differentiated. The *exterior derivative* d is an operator that turns k-forms into $k+1$-forms, according to very simple rules:

- The exterior derivative of a 0-form $f(\mathbf{x})$ in $\mathbb{R}^n$ (recall, f is just a function) is

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n. \tag{5}$$

- The exterior derivative of the k-form

$$\omega = \sum_{i_1,\ldots,i_k} F_{i_1,\ldots,i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

  is given by

$$d\omega = \sum_{i_1,\ldots,i_k} dF_{i_1,\ldots,i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \tag{6}$$

  where $dF_{i_1,\ldots,i_k}$ is computed according to equation (5).

  Example 10: Let $\omega = (x_1 + x_3^2) dx_1 \wedge dx_2$. Then

$$\begin{aligned}
d\omega &= d(x_1 + x_3^2) \wedge dx_1 \wedge dx_2 \\
&= dx_1 \wedge dx_1 \wedge dx_2 + 2x_3\, dx_3 \wedge dx_1 \wedge dx_2 \\
&= 2x_3\, dx_3 \wedge dx_1 \wedge dx_2 \\
&= -2x_3\, dx_1 \wedge dx_3 \wedge dx_2 \\
&= 2x_3\, dx_1 \wedge dx_2 \wedge dx_3
\end{aligned}$$

  Notice the term with two dx_1's is zero, and in the last two steps I just put $d\omega$ in "standard" form.

Generalized Stokes' Theorem

The generalized version of Stokes' Theorem embodies almost everything we've done in this course; the divergence theorem and the original Stokes' Theorem are special cases of this more general theorem. Before stating this theorem we need a few preliminary remarks.

A smooth k-manifold M in $\mathbb{R}^n$ may or may not be orientable. If it is orientable then it has two possible orientations, as we saw in examples above. It's not hard to believe (though we won't stop to prove) that the boundary ∂M of a k-manifold M is itself a manifold of dimension $k-1$. The boundary may or may not be orientable. We will assume that the manifolds that follow, as well as their boundaries, are orientable. Recall that in the divergence theorem we related a volume integral to a surface integral, and for things to work out properly we needed the surface (in particular, $d\mathbf{A}$) to be oriented properly. Similarly in Stokes' theorem, after choosing a direction for the unit normal $\mathbf{n}$ on a surface S, we then had to orient ∂S with the right hand rule. Similar considerations hold in higher dimensions. If we have a manifold M and we have chosen an orientation for M, then this induces a "consistent" orientation on ∂M, somewhat akin to the right hand rule. How this can be done we won't go into for now. The down side is that our integrals may occasionally be off by a minus sign.

Stokes' Theorem: (Generalized version) Let M be a smooth oriented k-manifold M with consistently oriented smooth boundary ∂M. Let ω be a $k-1$ form defined in a neighborhood of M. Then

$$\int_M d\omega = \int_{\partial M} \omega.$$

Example 11: Let M be as in Example 5. Given how M was parameterized, you will probably believe that ∂M can be described as the image of the boundary of the unit disk, $u_1^2 + u_2^2 = 1$, under the mapping from $\mathbb{R}^2$ to $\mathbb{R}^4$ that parameterized M. Given that the boundary of the disk can be parameterized by $u_1 = \cos(t), u_2 = \sin(t)$, it seems reasonable that the one dimensional curve ∂M can be parameterized as

$$\mathbf{Y}(t) = (\cos(t), \cos(t) - \sin(t), 3 - \cos(t) + \cos(t)\sin(t), -3\sin(t))$$

(notice I'm using $\mathbf{Y}$ for ∂M.) Now let $\omega = x_3^2\, dx_1$. It's easy to compute that $d\omega = -2x_3\, dx_1 \wedge dx_3$.

Let's first compute $\int_M d\omega$:

$$\begin{aligned}
\int_M d\omega &= \int_{-1}^{1}\int_{-\sqrt{1-u_1^2}}^{\sqrt{1-u_1^2}} d\omega\left(\frac{\partial X}{\partial u_1}, \frac{\partial X}{\partial u_2}\right) \\
&= \int_{-1}^{1}\int_{-\sqrt{1-u_1^2}}^{\sqrt{1-u_1^2}} -2x_3 \det\begin{bmatrix} 1 & 0 \\ u_2 - 1 & u_1 \end{bmatrix} du_2\, du_1 \\
&= \int_{-1}^{1}\int_{-\sqrt{1-u_1^2}}^{\sqrt{1-u_1^2}} -2(3 - u_1 + u_1u_2)u_1\, du_2\, du_1 = \frac{\pi}{2}.
\end{aligned}$$

Now for $\int_{\partial M} \omega$ we have

$$\begin{aligned}
\int_{\partial M} \omega &= \int_0^{2\pi} \omega(\mathbf{Y}'(t))\, dt \\
&= \int_0^{2\pi} (3 - \cos(t) + \cos(t)\sin(t))^2(-\sin(t))\, dt \\
&= \frac{\pi}{2}.
\end{aligned}$$

Example 12: Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ be a vector field in three dimensions and let $\omega = F_1 dx_1 + F_2 dx_2 + F_3 dx_3$, a 1-form. Suppose that M is an orientable 2-manifold (a surface) in three dimensions with one dimensional boundary ∂M. As we saw in Example 4,

$$\int_{\partial M} \omega = \int_{\partial M} \mathbf{F} \cdot d\mathbf{R}.$$

Now let's compute $d\omega$ and integrate it over M. If you write out $d\omega$ you find that many terms cancel (they contain a wedge product with identical dx_i's) and you obtain

$$\begin{aligned} d\omega &= \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) dx_1 \wedge dx_2 + \left(\frac{\partial F_3}{\partial x_1} - \frac{\partial F_1}{\partial x_3}\right) dx_1 \wedge dx_3 \\ &+ \left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}\right) dx_2 \wedge dx_3 \end{aligned}$$

Now define a vector field $\mathbf{G} = G_1\mathbf{i} + G_2\mathbf{j} + G_3\mathbf{k}$ by taking

$$\begin{aligned} G_1 &= \left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}\right) \\ G_2 &= \left(\frac{\partial F_3}{\partial x_1} - \frac{\partial F_1}{\partial x_3}\right) \\ G_3 &= \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right). \end{aligned}$$

In short, $\mathbf{G} = \nabla \times \mathbf{F}$. Then we can write

$$d\omega = G_1 dx_2 \wedge dx_3 + G_2 dx_1 \wedge dx_3 + G_3 dx_1 \wedge dx_2.$$

If you look at Example 6 (and replace $\mathbf{F}$ there by $\mathbf{G}$ here), you find that

$$\int_M d\omega = \int_M G \cdot d\mathbf{A}.$$

But given that $\mathbf{G} = \nabla \times \mathbf{F}$, we conclude from the generalized Stokes' Theorem that

$$\int_{\partial M} \mathbf{F} \cdot d\mathbf{R} = \int_M (\nabla \times \mathbf{F}) \cdot d\mathbf{A}$$

which is the usual version of Stokes' Theorem in three dimensions!

Example 12: Let M be a bounded region in three dimensions with orientable two dimensional boundary ∂M. Again, let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ be a smooth vector field defined on M and define the 2-form

$$\omega = F_1\, dx_2 \wedge dx_3 + F_2\, dx_1 \wedge dx_3 + F_3\, dx_1 \wedge dx_2$$

Again referring back to Example 6, we find that

$$\int_{\partial M} \omega = \int_{\partial M} \mathbf{F} \cdot d\mathbf{A},$$

the flux of $\mathbf{F}$ over ∂M. Now compute $d\omega$. Most of the terms are zero and we end up with

$$d\omega = \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3} \right) dx_1 \wedge dx_2 \wedge dx_3 = (\nabla \cdot \mathbf{F}) dx_1 \wedge dx_2 \wedge dx_3.$$

Now from Example 8 we know that (if M is oriented correctly)

$$\int_M d\omega = \int_M \nabla \cdot \mathbf{F} \, dV.$$

From the general Stokes' Theorem we conclude that

$$\int_{\partial M} \mathbf{F} \cdot d\mathbf{A} = \int_M \nabla \cdot \mathbf{F} \, dV.$$

which is the divergence theorem!