# HOMEWORK 1

DAN YASAKI

1. Read Gunnells *Modular forms TWIGS*.
   `http://www.math.umass.edu/~gunnells/talks/modforms.pdf`
2. Read Chapter 1 of textbook.
3. §1.6 (1.1) Note that this shows the action of $\mathrm{GL}_2(\mathbb{R})$ preserves the complex upper halfplane.
   **Solution:** Let $z = x + iy \in \mathbb{C}$ with $y > 0$, and let $a, b, c, d \in \mathbb{R}$ with $ad - bc > 0$. We want to show that
   $$\mathrm{Im}\left(\frac{az+b}{cz+d}\right) > 0.$$
   Multiply the numerator and denominator by $\overline{cz+d} = c\bar{z} + d$ to get
   $$\begin{aligned}\left(\frac{az+b}{cz+d}\right) &= \left(\frac{az+b}{cz+d}\right)\left(\frac{c\bar{z}+d}{\overline{cz+d}}\right)\\ &= \frac{ac|z|^2 + bc\bar{z} + adz + bd}{|cz+d|^2}\\ &= \frac{ac|z|^2 + bcx - bciy + adx + adiy + bd}{|cz+d|^2}.\end{aligned}$$
   The imaginary part is $\frac{(ad-bc)y}{|cz+d|^2}$, which is greater than 0 as desired.
4. §1.6 (1.3)
   **Solution:** Recall a weakly modular function is a meromorphic function such that for all $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$,
   $$f(\gamma \cdot z) = (cz+d)^k f(z).$$
   (a) Suppose $f$ and $g$ are weakly modular functions of weight $k_1$ and $k_2$, respectively. We want to show the product $h = fg$ is a weakly modular function. The product of meromorphic functions is meromorphic, so it suffices to show that $h$ satisfies the correct equivariance properties. We compute
   $$\begin{aligned}h(\gamma \cdot z) &= f(\gamma \cdot z)g(\gamma \cdot z)\\ &= (cz+d)^{k_1} f(z)(cz+d)^{k_2} g(z)\\ &= (cz+d)^{k_1+k_2} f(z)g(z)\\ &= (cz+d)^{k_1+k_2} h(z).\end{aligned}$$
   (b) Suppose $f$ is a weakly modular function of weight $k$. We want to show that $1/f$ is a weakly modular function. The reciprocal of a meromorphic function is meromorphic, so it suffices to show that $h = 1/f$ satisfies the correct equivariance

properties. We compute

$$\begin{aligned} h(\gamma \cdot z) &= \frac{1}{f(\gamma \cdot z)} \\ &= \frac{1}{(cz+d)^k f(z)} \\ &= (cz+d)^{-k} \frac{1}{f(z)} \\ &= (cz+d)^{-k} h(z). \end{aligned}$$

(c) Suppose $f$ and $g$ are modular functions. We want to show that $fg$ is a modular function. Recall that a modular funciton is a weakly modular function that is meromorphic at infinity. Above we show that the product of weakly modular functions is weakly modular, so it suffices to show that $h = fg$ is meromorphic at infinity, assuming $f$ and $g$ are meromorphic at infinity. This can be shown by multiplying the respective $q$ expansions. Specifically, let

$$f(z) = \sum_{n \geq m_1}^{\infty} a_n q^n \quad \text{and} \quad g(z) = \sum_{n \geq m_2}^{\infty} b_n q^n.$$

Then the $q$-expansion of $h$ is

$$\begin{aligned} h(z) &= \left( \sum_{n \geq m_1}^{\infty} a_n q^n \right) \left( \sum_{n \geq m_2}^{\infty} b_n q^n \right) \\ &= a_{m_1} b_{m_2} q^{m_1 + m_2} + \cdots \end{aligned}$$

Since $m_1 + m_2 \in \mathbb{Z}$, it follows that $h$ is meromorphic at infinity.

(d) Suppose $f$ and $g$ are modular forms. We want to show that $h = fg$ is a modular form. Recall that a modular form is a modular function that is homolorphic on $\mathfrak{h}$ and holomorphic at infinity. We show above that the product of modular functions is a modular function. The product of holomorphic functions is holomorphic. Thus it suffices to show that $h$ is holomorphic at infinity assuming $f$ and $g$ are holomorphic at infinity. As above, we just look at the $q$-expansions. Specifically, let

$$f(z) = \sum_{n \geq 0}^{\infty} a_n q^n \quad \text{and} \quad g(z) = \sum_{n \geq 0}^{\infty} b_n q^n.$$

Then the $q$-expansion of $h$ is

$$\begin{aligned} h(z) &= \left( \sum_{n \geq 0}^{\infty} a_n q^n \right) \left( \sum_{n \geq 0}^{\infty} b_n q^n \right) \\ &= a_0 b_0 + (a_1 b_0 + a_0 b_1) q + \cdots \end{aligned}$$

Since $m_1 + m_2 \in \mathbb{Z}$, it follows that $h$ is meromorphic at infinity.

5. §1.6 (1.4)

**Solution:** Recall

$$\Gamma_1(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \right\}.$$

(a) Let $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $h = \begin{bmatrix} q & r \\ s & t \end{bmatrix}$ be elements of $\Gamma_1(N)$. Then

$$a \equiv d \equiv q \equiv t \equiv 1 \pmod{N}$$

and

$$c \equiv s \equiv 0 \pmod{N}.$$

It follows that

$$g^{-1} = \begin{bmatrix} d & -b \\ -ca & \end{bmatrix} \in \Gamma_1(N).$$

We compute

$$gh = \begin{bmatrix} aq + bs & ar + bt \\ qc + ds & cr + dt \end{bmatrix}.$$

Since $c \equiv s \equiv 0 \pmod{N}$, we have $qc + ds \equiv 0 \pmod{N}$. Since $a \equiv q \equiv 1 \pmod{N}$ and $s \equiv 0 \pmod{N}$, we have $aq + bs \equiv 1 \pmod{N}$. Similarly, we have $cr + dt \equiv 1 \pmod{N}$. Thus $gh \in \Gamma_1(N)$, and $\Gamma_1(N)$ is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

(b) We want to prove that $\Gamma_1(N)$ has finite index in $\mathrm{SL}_2(\mathbb{Z})$, where

$$\Gamma(N) = \ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})).$$

First note that $\Gamma(N) \subset \Gamma_1(N)$. It follows that

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(N)] \le [\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N)] \le \#\,\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) < \infty.$$

(c) We want to prove that $\Gamma_0(N)$ has finite index in $\mathrm{SL}_2(\mathbb{Z})$. This follows because $\Gamma_1(N) \subset \Gamma_0(N)$, and we show above that $\Gamma_1(N)$ has finite index in $\mathrm{SL}_2(\mathbb{Z})$.

(d) We want to prove that $\Gamma_0(N)$ and $\Gamma_1(N)$ have level $N$. Recall that the level of a congruence subgroup is the smallest positive integer $n$ such that the congruence subgroup contains $\Gamma(n)$. Let $t < N$. Then $g = \begin{bmatrix} 1 & 0 \\ t & 0 \end{bmatrix} \in \Gamma(t)$, and $g \notin \Gamma_1(N)$ and $g \notin \Gamma_0(N)$. It follows that the level of $\Gamma_1(N)$ and the level of $\Gamma_0(N)$ is greater than or equal to $N$. It is clear that $\Gamma(N) \subset \Gamma_0(N)$ and $\Gamma(N) \subset \Gamma_0(N)$, and so the level is less than or equal to $N$. It follows that the level is exactly $N$.

6. §1.6 (1.7) Note that this shows that

$$(f^{[\gamma]_k})(z) = \det(\gamma)^{k-1}(cz + d)^{-k} f(\gamma(z))$$

defines a right action of $\mathrm{GL}_2(\mathbb{R})$ on the set of functions $f \colon \mathfrak{h}^* \to \mathbb{C}$.

**Solution:** For $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, let $j$ be the automorphy factor $j(\gamma, z) = (cz + d)$. Note that

$$f^{[\gamma]_k}(z) = \det(\gamma)^{k-1} j(\gamma, z)^{-k} f(\gamma \cdot z).$$

We want to show that

$$f^{[\gamma_1\gamma_2]_k}(z) = ((f^{[\gamma_1]_k})^{[\gamma_2]_k})(z).$$

The left side is

$$\det(\gamma_1\gamma_2)^{k-1} j(\gamma_1\gamma_2, z)^{-k} f((\gamma_1\gamma_2) \cdot z)$$

and the right side is

$$\det(\gamma_2)^{k-1} \det(\gamma_1)^{k-1} j(\gamma_1, \gamma_2 \cdot z)^{-k} j(\gamma_2, z)^{-k} f(\gamma_1 \cdot (\gamma_2 \cdot z)).$$

Thus it suffices to show that

(a) $(\gamma_1\gamma_2)\cdot z = \gamma_1\cdot(\gamma_2\cdot z)$ and
(b) $j(\gamma_1\gamma_2, z) = j(\gamma_1, \gamma_2\cdot z)j(\gamma_2, z)$.

Consider the vector $\begin{bmatrix} z \\ 1 \end{bmatrix}$. Then one can relate the action of matrices on the upper half plane with the regular matrix multiplication on vectors by

$$\gamma \begin{bmatrix} z \\ 1 \end{bmatrix} = \begin{bmatrix} \gamma\cdot z \\ 1 \end{bmatrix} j(\gamma, z).$$

It follows that

$$(\gamma_1\gamma_2)\begin{bmatrix} z \\ 1 \end{bmatrix} = \gamma_1 \begin{bmatrix} \gamma_2\cdot z \\ 1 \end{bmatrix} j(\gamma_2, z) \tag{1}$$

$$= \begin{bmatrix} \gamma_1\cdot(\gamma_2\cdot z) \\ 1 \end{bmatrix} j(\gamma_1, \gamma_2\cdot z)j(\gamma_2, z). \tag{2}$$

On the other hand,

$$(\gamma_1\gamma_2)\begin{bmatrix} z \\ 1 \end{bmatrix} = \begin{bmatrix} (\gamma_1\gamma_2)\cdot z \\ 1 \end{bmatrix} j(\gamma_1\gamma_2, z). \tag{3}$$

Setting (2) equal to (3) gives the desired result.

Dan Yasaki, Department of Mathematics and Statistics, University of North Carolina at Greensboro, Greensboro, NC 27402-6170, USA
*E-mail address*: d_yasaki@uncg.edu