# HOMEWORK 2

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The problems are from the text page 33.

1. Read Chapter 2.1 - 2.4 of textbook.
2. 2.1
   **Solution:** By Proposition 2.8, we have that
   $$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k} = -\frac{(2\pi i)^k}{2 \cdot k!} \cdot B_k.$$
   Since $B_{26} = \frac{8553103}{6}$, it follows that
   $$\sum_{n=1}^{\infty} \frac{1}{n^{26}} = -\frac{(2\pi i)^{26}}{2 \cdot 26!} \cdot \frac{8553103}{6} = \frac{1315862\pi^{26}}{11094481976030578125}.$$
3. 2.3 Do this problem 2 ways. First compute the terms using (2.1.3). Second, check your answer using the Sage $q$-expansion of $E_8$.
   **Solution:** According to Sage, we have that
   $$E_8 = \frac{1}{480} + q + 129q^2 + 2188q^3 + O(q^4).$$
   Let us verify this using (2.1.3), which states that
   $$E_k = -\frac{B_k}{2k} + q + \sum_{n=2}^{\infty} \sigma_{k-1}(n) q^n.$$
   Since $B_8 = -\frac{1}{30}$, we have
   $$a_0 = -\frac{1}{30 \cdot (2 \cdot 8)} = -\frac{1}{480}.$$
   We see that $a_1 = 1$ matches as well. We compute
   $$a_2 = \sigma_7(2) = 1^7 + 2^7 = 129$$
   $$a_3 = \sigma_7(3) = 1^7 + 3^7 = 2188.$$
4. 2.6
   **Solution:** We want to express
   $$f = q^2 + 192q^3 - 8280q^4 + \cdots \in S_{28}(\mathrm{SL}_2(\mathbb{Z}))$$

as a polynomial in $E_4$ and $E_6$. We know that $M_{28}(\mathrm{SL}_2(\mathbb{Z}))$ has dimension $\lfloor \frac{28}{12} \rfloor + 1 = 3$. We can take as a basis

$$\begin{aligned} b_1 = E_4^7 &= \frac{1}{45864714240000000} + \frac{7}{191102976000000}q + \frac{7}{262144000000}q^2 \\ &\quad + \frac{526729}{47775744000000}q^3 + \frac{538962991}{191102976000000}q^4 + O(q^5) \\ b_2 = E_4^4 E_6^2 &= \frac{1}{842764124160000} - \frac{1}{17557585920000}q - \frac{101}{216760320000}q^2 \\ &\quad - \frac{349423}{4389396480000}q^3 + \frac{776649707}{17557585920000}q^4 + O(q^5) \\ b_3 = E_4 E_6^4 &= \frac{1}{15485790781440} - \frac{37}{322620641280}q + \frac{251}{3982970880}q^2 \\ &\quad - \frac{49501}{11522165760}q^3 - \frac{1656597781}{322620641280}q^4 + O(q^5). \end{aligned}$$

Let $B$ be the matrix whose columns are the Fourier coefficients of the $b_i$. Then we solve

$$Bx = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 192 \\ -8280 \end{bmatrix}$$

to express $f$ in terms of $b_i$ to find that

$$15360000000 b_1 - 564480000 b_2 + 5186160 b_3.$$

5. 2.7

**Solution:** It is not always the case that the Eisenstien series, normalized so that the constant coefficient is 1 lies is $\mathbb{Z}[[q]]$. The easiest way to see this is to use Sage to write out the $q$-expansions of the first few Eisenstein series and normalize as described. (Helpful hints: Remember that $k \geq 4$. To get the constant coefficient, evaluate $E_k(0)$.) When we do that, we see

$$\frac{1}{E_{12}(0)} \cdot E_{12} = 1 + \frac{65520}{691}q + \frac{134250480}{691}q^2 + \frac{11606736960}{691}q^3 + O(q^4).$$

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