HOMEWORK 5

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1. Improve your `cone_containing` function as follows. Sometimes the point is not on the interior of the triangular cone, but on the boundary. In this case, instead of returning the triple of vertices defining the triangle, return the pair of vertices defining the edge.
2. Double-check your functions created so far to account for the WARNING on 4 on the section Actions of the Notes. Specifically, the `reduce` command may not terminate if you do not.
3. Let $\mathbf{u}_0$ denote the unimodular symbol $\{0, \infty\}$. In your notations, this corresponds to $\left[\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right]$. Compute the stabilizer G_0 of $\mathbf{u}_0$ in $\mathrm{SL}_2(\mathbb{Z})$. Specifically, find all the $g \in \mathrm{SL}_2(\mathbb{Z})$ such that
$$g \cdot \mathbf{u}_0 = \pm \mathbf{u}_0.$$
4. Read through the Sage docs modular symbols to see what we are trying to duplicate. Also, they have some functions for dealing with the projective line over $\mathbb{Z}/N\mathbb{Z}$ that may be useful. You can access the documentation through sage by typing `P1List??` and `ModularSymbols??` If you prefer, see relevant section of `http://www.sagemath.org/doc/reference/modsym.html` The `ModularSymbols` class has everything we are trying to duplicate, and should be used to check your answers below.
5. We can use `P1List` to fix an ordering on the elements of $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$. The group G_0 acts on $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$ on the right. Write a function `one_cell_orbits` that takes as input a positive integer N, and returns a list $[O_1, \cdots O_k]$, where each O_i is a G_0-orbit of $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$. In other words, each O_i has the form
$$O_i = [v \cdot g \ : \ g \in G_0],$$
for some $v \in \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$.
6. Let `cycles(N)` denote the $\mathbb{Q}$-vector space spanned by G_0-orbit of $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$. In other words, `cycles(N)` is a rational vector space of dimension equal to the cardinality of `one_cell_orbits(N)`. Let `boundaries(N)` denote the space spanned by
$$\{r \cdot \{0, 1\} + r \cdot \{1, \infty\} + r \cdot \{\infty, 0\}\},$$
where r ranges over coset representatives for $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbb{Z})$, viewed as a subspace of `cycles(N)`. Verify, in several examples, that the quotient `cycles(N)/boundaries(N)` is isomorphic to the space of modular symbols of level N.

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