NOTES AND QUESTIONS FOR MODULAR SYMBOLS AND HOMOLOGY

DAN YASAKI

Abstract. Notes and questions about modular symbols and homology. This arose in the context of a summer reading course from Stein's [1].

Let $X_0(N) = \Gamma_0(N)\backslash\mathfrak{h}^*$, where $\mathfrak{h}^* = \mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q})$. Then $X_0(N)$ is a compact, orientable Riemann surface (1-dimensional complex manifold). It follows that $X_0(N)$ is a 2-dimensional, orientable, compact, real manifold. Thus $X_0(N)$ is a g-holed torus, where g is called the *genus* of $X_0(N)$.

1. Homology, forms, and pairings

Exercise 1. Let $f \in S_2(\Gamma_0(N))$. Prove that $f(z)dz$ defines a holomorphic differential 1-form on $X_0(N)$. Since f is a cusp form, it vanishes at every cusp for $\Gamma_0(N)$. Thus it suffices to prove that

$$f(z)dz = f(\gamma \cdot z)d(\gamma \cdot z)$$

for every $\gamma \in \Gamma_0(N)$.

Note that $f(z)dz$ is defined to be an object for $\mathfrak{h}^*$. The fact that it is invariant under the action of $\Gamma_0(N)$ is what tells us that it defines an object that makes sense on $X_0(N)$.

Let

$$\langle \cdot, \cdot \rangle \colon S_2(\Gamma_0(N)) \times H_1(X_0(N), \mathbb{Z}) \to \mathbb{C}$$

be the pairing defined by integration. Specifically, for a path $x \subset X_0(N)$ and a cusp form $f \in S_2(\Gamma_0(N))$,

$$\langle f, x \rangle = 2\pi i \int_x f(z)dz. \tag{1}$$

Exercise 2. Our previous worksheet on sheet on pairings had $\langle \cdot, \cdot \rangle \colon M \times N \to L$, where M, N, L were all vector spaces over a field R. Here, we must relax this a bit and consider R-modules, where R is just a ring. What is M, N, L, R for the pairing defined in (1)?

Exercise 3. Explain why (1) is well defined as a pairing of $S_2(\Gamma_0(N))$ and $H_1(X_0(N), \mathbb{Z})$. Specifically, (1) is defined for paths. If we take 2 paths x_1, x_2 in $\mathfrak{h}^*$ which are in the same homology class in $H_1(X_0(N), \mathbb{Z})$, how do we know that $\langle f, x_1 \rangle = \langle f, x_2 \rangle$ for all $f \in S_2(x_0(N))$?

Exercise 4. Use (2) to induce a pairing

$$\langle \cdot, \cdot \rangle \colon S_2(\Gamma_0(N) \times H_1(X_0(N), \mathbb{R}) \to \mathbb{C}. \tag{2}$$

There is a action of Hecke operators on $H_1(X_0(N), \mathbb{Z})$, which induces an action on $H_1(X_0(N), \mathbb{R})$, which we will talk about more later.

Theorem 1 ([1, Theorem 3.4]). *The pairing* (2) *is a perfect pairing and Hecke equivariant in the sense that*

$$\langle fT_n, x\rangle = \langle f, T_n x\rangle,$$

for each Hecke operator T_n.

If M is a g-holed torus, then the genus of M is g, and $H_1(M, \mathbb{Z})$ is a free abelian group of rank $2g$. It follows that $H_1(M, \mathbb{R})$ is a $2g$-dimensional (real) vector space.

Exercise 5. Prove that $\dim_{\mathbb{C}}(S_2(\Gamma_0(N)) = g$, where g is the genus of $X_0(N)$. Hint: Use the perfect pairing to get isomorphisms between various spaces. Then compute dimensions and solve.

2. Modular symbols

Let $\mathbb{M}_2$ be the free abelian group generated by

$$\{\{\alpha, \beta\} \mid \alpha, \beta \in \mathbb{P}^1(\mathbb{Q})\},$$

modulo the 3-term relations

$$\{\alpha, \beta\} + \{\beta, \gamma\} + \{\gamma, \alpha\} = 0$$

and torsion.

Note that the element $\{\alpha, \beta\}$ is NOT a set. It is just notation for an object, so order matters.

Exercise 6. Since order matters, perhaps we can think of $\{\alpha, \beta\}$ as a directed edge joining α and β. Use this idea to draw what the 3-term relation says.

Exercise 7. Prove that for all $\alpha, \beta \in \mathbb{P}^1(\mathbb{Q})$,

(1) $\{\alpha, \alpha\} = 0$,
(2) $\{\alpha, \beta\} = -\{\beta, \alpha\}$, and
(3) $\{\alpha, \beta\} = \{\alpha, 0\} + \{0, \beta\}$.

Exercise 8. Define a left action of $\mathrm{GL}_2(\mathbb{Q})$ on $\{\alpha, \beta\} \in \mathbb{M}_2$ by

$$g \cdot \{\alpha, \beta\} = \{g \cdot \alpha, g \cdot \beta\}.$$

Explain this action.

Exercise 9. The exercise above how $\mathrm{GL}_2(\mathbb{Q})$ acts on certain elements in $\mathbb{M}_2$. Explain how to extend the action to all of $\mathbb{M}_2$.

Definition 2. The *modular symbols for* $\Gamma_0(N)$, denoted $\mathbb{M}_2(\Gamma_0(N))$ is the quotient of $\mathbb{M}_2$ modulo the relations $x - g \cdot x = 0$ for all $x \in \mathbb{M}_2$ and $g \in \Gamma_0(N)$ and modulo any torsion.

Exercise 10. Let $n, m \in \mathbb{Z}$. Prove that $\{n, m\} = 0$ in $\mathbb{M}_2(\Gamma_0(N))$. Hint: See [1, Example 3.6].

Exercise 11. For $g, h \in \Gamma_0(N)$, prove that

$$\{\alpha, gh \cdot \alpha\} = \{\alpha, g \cdot \alpha\} + \{\alpha, h \cdot \alpha\}$$

in $\mathbb{M}_2(\Gamma_0(N))$. Hint: You will need the 3-term relations combined with the fact that $x = g \cdot x$ for all $x \in \mathbb{M}_2(\Gamma_0(N))$ and $g \in \Gamma_0(N)$.

Exercise 12. For $\alpha, \beta \in \mathbb{P}^1(\mathbb{Q})$ and $g \in \Gamma_0(N)$, prove that $\{\alpha, g \cdot \alpha\} = \{\beta, g \cdot \beta\}$ in $\mathbb{M}_2(\Gamma_0(N))$. Hint: Notice that

$$\{\alpha, g \cdot \alpha\} = \{\alpha, \beta\} + \{\beta, g \cdot \beta\} + \{g \cdot \beta, g \cdot \alpha\}.$$

References

[1] W. Stein, *Modular forms, a computational approach*, Graduate Studies in Mathematics, vol. 79, American Mathematical Society, Providence, RI, 2007, With an appendix by Paul E. Gunnells.

Dan Yasaki, Department of Mathematics and Statistics, The University of North Carolina at Greensboro, Greensboro, NC 27412, USA

E-mail address: d_yasaki@uncg.edu

URL: http://www.uncg.edu/~d_yasaki/