The Dual-Wind Discontinuous Galerkin Method

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SIAM Central States Sectional Meeting
Spring 2015

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Supported in part by NSF

Introduction

PDE Problem:

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega \subset \mathbb{R}^d, \\ u &= g \qquad \text{on } \partial\Omega. \end{aligned}$$

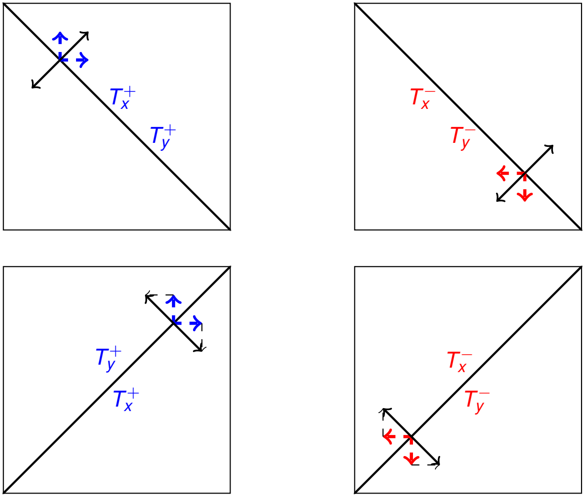
Goal: Develop an optimally convergent DG method that:

- is symmetric when written in primal form
- naturally enforces BCs without boundary penalization terms
- does not require interior stabilization

All of the following has been extended for Neumann BCs and the biharmonic equation with Lagrange basis functions.

DG Derivative Operators

Interior Trace Values



THE DWDG METHOD

Suppose that $\gamma_{\min} > 0$. Let $u_h \in V_{h,r}$ be the unique solution for the DWDG method and $u \in H^{s+1}(\Omega)$ be the PDE solution with $1 \leq s \leq r$. Then there holds

$$\|u - u_h\|_{1,h} \leq C\left(\sqrt{\gamma_{\max}} + \frac{1}{\sqrt{\gamma_{\min}}}\right) h^s |u|_{H^{s+1}(\Omega)}.$$

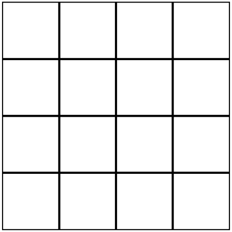
Moreover, if the triangulation is quasi-uniform, there exists a constant $C_* > 0$ such that, for $\gamma_{\min} > -C_*$, there holds

$$\|u - u_h\|_{1,h} \leq C\left(\sqrt{|\gamma_{\max}|} + \frac{1}{\sqrt{\gamma_{\min} + C_*}}\right) h^s |u|_{H^{s+1}(\Omega)}.$$

Numerical Tests

Test 3: $u = x^3 - 2xy^2 + xy - y^2 + y - 3$ with a Cartesian mesh

	$h_x = h_y$	$\|u - u_h\|_{L^2}$	rate	$\|\nabla u - \nabla_h^+ u_h\|_{L^2}$	rate	$\|\nabla u - \nabla_h^- u_h\|_{L^2}$	rate
$r = 0$	1/4	1.21E+00		3.01E+00		2.98E+00	
	1/8	6.32E-01	0.93	1.83E+00	0.72	1.86E+00	0.68
	1/16	3.07E-01	1.04	1.03E+00	0.84	1.06E+00	0.82
	1/32	1.48E-01	1.06	5.55E-01	0.89	5.73E-01	0.88
$r = 1$	1/4	1.81E-01		7.18E-01		7.89E-01	
	1/8	4.42E-02	2.04	4.88E-01	0.56	4.96E-01	0.67
	1/16	1.09E-02	2.02	2.86E-01	0.77	2.87E-01	0.79
	1/32	2.73E-03	1.99	1.54E-01	0.89	1.55E-01	0.89
$r = 2$	1/4	1.31E-02		1.39E-01		1.39E-01	
	1/8	1.56E-03	3.07	4.03E-02	1.78	4.03E-02	1.78
	1/16	1.92E-04	3.03	1.07E-02	1.92	1.07E-02	1.92
	1/32	2.37E-05	3.01	2.75E-03	1.96	2.75E-03	1.96
$r = 3$	1/8	1.35E-10		2.23E-09		2.18E-09	



Conclusion

- We propose a symmetric DG method for Poisson's equation that is well-posed and exhibits optimal convergence rates without interior or boundary penalization. Dirichlet (and Neumann) boundary data appears naturally in the scheme.
- For piecewise constants on a Cartesian mesh, the method is equivalent to the standard second order Finite Difference method.
- The method is based on combining and composing various DG FE derivative operators similar to the construction of FD methods.
- The method utilizes multiple trace values by incorporating 2 discrete derivatives that both incorporate d trace values.
- The method can be extended for other second order elliptic problems, different boundary conditions, and higher order problems.

REFERENCES

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Thank you for your attention!