Calculus I to A Transition Test

Review **Solutions**

Try to solve the problems before looking at the solutions. Check your answers with the solutions. Learn from you mistakes.

Do not just study the answers.

Questions that are asked in a sentence should be answered with a sentence.

1. Define the *antiderivative* of a function f on an interval $[a,b]$.

   Solution. A function F is an *antiderivative of f on* $[a,b]$ if $F'(x) = f(x)$ for all x in $[a,b]$.

2. Find functions f and g with

   $$\int f(x) \cdot g(x)\, dx \neq \int f(x)\, dx \cdot \int g(x)\, dx.$$

   Solution. Let $f(x) = g(x) = x$. Then $\displaystyle\int f(x) \cdot g(x)\, dx = \int x^2\, dx = \frac{1}{3}x^3$ and $\displaystyle\int f(x)dx = \int g(x)\, dx = \int x\, dx = \frac{1}{2}x^2$. Thus $\displaystyle\int f(x) \cdot g(x)\, dx = \frac{1}{3}x^3 \neq \int f(x)\, dx \cdot \int g(x)\, dx = \frac{1}{4}x^4$.

3. Solve the initial value problem $\frac{ds}{dt} = 12t(3t^2 - 1)$, $s(1) = 3$.

   Solution. Taking the antiderivative of both sides of the differential equation yields:

   $$s(t) = \int 36t^3 - 12t\, dt = \frac{36}{4}t^4 - \frac{12}{2}t^2 + c = 9t^4 - 6t^2 + c$$

   We use the intial value $s(1) = 3$ to find c: $s(1) = 9 - 6 + c = 3$, so $c = 0$. Thus $s(t) = 9t^4 - 6t^2$.

4. Solve the initial value problem $\frac{d^2y}{dx^2} = \frac{3x}{8}$, $y'(4) = 3$, $y(4) = 5$.

Solution. $y'(x) = \frac{dy}{dx} = \int \frac{3x}{8} dx = \frac{3x^2}{16} + c_1$ and $y'(4) = \frac{3 \cdot 4^2}{16} + c_1 = 3 + c_1 = 3$, so $y'(x) = \frac{3x^2}{16}$. $y(x) = \int \frac{3x^2}{16} dx = \frac{x^3}{16} + c_2$ and $y(4) = \frac{4^3}{16} + c_2 = 4 + c_2 = 5$, thus $c_2 = 1$. The solution of the initial value problem is $y(x) = \frac{x^3}{16} + 1$.

5. Find $\displaystyle\int 24x^3 - 20x^4 + 25x^5 + 9\, dx$.

Solution. $\displaystyle\int 24x^3 - 20x^4 + 25x^5 + 9\, dx = 6x^4 - 4x^5 + \frac{25}{6}x^6 + 9x + c.$

6. Find $\displaystyle\int x^2 - 16x^3 + \frac{1}{3x^2}\, dx$.

Solution. $\displaystyle\int x^2 - 16x^3 + \frac{1}{3x^2}\, dx = \frac{1}{3}x^3 - 4x^4 - \frac{1}{3}\frac{1}{x} + c.$

7. Find $\displaystyle\int \frac{8x^8 + 6x^6 - 80}{x^6}\, dx$.

Solution. $\displaystyle\int \frac{8x^8 + 6x^6 - 80}{x^6}\, dx = \int 8x^2 + 6 - \frac{80}{x^6}\, dx = \frac{8}{3}x^3 + 6x + \frac{80}{5x^5} + c$

8. Find $\displaystyle\int 2e^x - 3e^{-2x}\, dx$.

Solution. $\displaystyle\int 2e^x - 3e^{-2x}\, dx = 2e^x + \frac{3}{2}e^{-2x} + c$

9. Find $\displaystyle\int \frac{2}{\sqrt{x}}\, dx$.

Solution. $\displaystyle\int \frac{2}{\sqrt{x}}\, dx = \int 2x^{-1/2}\, dx = 4x^{1/2} = 4\sqrt{x}.$

10. Find $\displaystyle\int -\sin(x)\, dx$.

Solution. $\displaystyle\int -\sin(x)\, dx = \cos(x) + c.$

11. Using rectangles whose height is given by the value of the function at the midpoint of the rectangle's base estimate the area under the graph of $f(x) = 8/x$ between $x = 1$ and $x = 33$ using

    (a) 2 rectangles of equal width

    Solution. $f(9) \cdot 16 + f(25) \cdot 16 \approx 19.3422$ is an estimate for the area under the graph using 2 rectangles.

    (b) 4 rectangles of equal width

    Solution. $f(5) \cdot 8 + f(13) \cdot 8 + f(21) \cdot 8 + f(29) \cdot 8 \approx 22.9776$ is an estimate for the area under the graph using 4 rectangles.

    Round to four decimal places.

12. Let a be a positive real number. What is the area under the graph of $f(x) = 5$ between $x = 0$ and $x = a$?

    Solution. The area is $5a$.

13. Let a be a positive real number. What is the area under the graph of $f(x) = 5x$ between $x = 0$ and $x = a$?

    Solution. The area is the are of the triangle with vertices $(0, 0)$, $(a, 0)$, and $(a, 5a)$. The area is $5a^2/2$.

14. What is the area under the graph of $f(x) = 5x$ between $x = 1$ and $x = 7$?

    Solution. The area is the are of the triangle with vertices $(0, 0)$, $(7, 0)$, and $(7, 35)$ minus the area of the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 7)$. So The area is $7(5 \cdot 7)/2 - 1(5 \cdot 1)/2 = 120$.

15. Evaluate $\displaystyle\sum_{i=1}^{100} 3$.

    Solution. $\displaystyle\sum_{i=1}^{100} 3 = 100 \cdot 3 = 300$

16. Evaluate $\displaystyle\sum_{i=0}^{10} 2$.

Solution. $\sum_{i=1}^{10} 2 = 11 \cdot 2 = 22$

17. Evaluate $\sum_{i=0}^{999} 2i$.

Solution. $\sum_{i=0}^{999} 2i = \sum_{i=1}^{999} 2i = 2\frac{999(999+1)}{2} = 999000$

18. Evaluate $\sum_{k=0}^{3} \frac{k^2}{k+1}$.

Solution. $\sum_{k=0}^{3} \frac{k^2}{k+1} = 0 + \frac{1}{2} + \frac{4}{3} + \frac{9}{4} = \frac{49}{12}$

19. Evaluate $\sum_{k=1}^{9} \left(k^3 + 2\right)$.

Solution.

$$\sum_{k=1}^{9} k^3 + 2 = \left(\frac{9(9+1)}{2}\right)^2 + 18 = \frac{8100}{4} + 18 = 2043$$

20. Let $f(x) = x^3$.

(a) Give a formula for the approximation of the area under the curve f between $x = 0$ and $x = 2$ in sigma notation. For your approximation use n rectangles of equal width and use the right endpoint for the height.

Solution. The area of the i-th rectangle is $\left(\frac{2i}{n}\right)^3 \frac{2}{n}$. The area under the curve is approximated by

$$\sum_{i=1}^{n} \left(\frac{2i}{n}\right)^3 \frac{2}{n} = 2^3 \left(\frac{1}{n}\right)^3 \frac{2}{n} \sum_{i=1}^{n} i^3 = 16 \left(\frac{1}{n}\right)^4 \left(\frac{n(n+1)}{2}\right)^2 = 4\frac{(n+1)^2}{n^2}.$$

(b) Take the limit of your formula as $n \to \infty$.

Solution. $\lim_{n\to\infty} 4\frac{(n+1)^2}{n^2} = 4$

(c) What is the exact area under the curve ?

Solution. The area under the curve is 4.

21. Give the definition of *Riemann sum* for a function f on an interval $[a, b]$.

Solution. Let
$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$
be a partition of $[a, b]$. Let $\Delta x_k = x_k - x_{k-1}$ be the width of the k^{th} subinterval. In each subinterval, pick a point x_k^*. Then the sum
$$S_P = \sum_{k=1}^{n} f(x_k^*)\ \Delta x_k$$
is called a *Riemann sum for f on* $[a, b]$.

22. Define the definite integral of a function f on the interval $[a, b]$.

Solution. Let $f(x)$ be a function defined on a closed interval$[a, b]$. We say that a number J is the *definite integral of f over* $[a, b]$ *and J is the limit of the Riemann sum* $\sum f(x_k^*)\Delta x_k$ *if the following statement is true.*

Given $\epsilon > 0$ there is a $\delta > 0$ such that for every partition P with $\|P\| < \delta$ and any choice of x_k^* in the k^{th} subinterval,
$$\left| \sum_{k=1}^{n} f(x_k^*)\Delta x_k - J \right| < \epsilon.$$

When the definite integral of f on $[a, b]$ *exists, we say that f is* integrable on $[a, b]$.

23. Express the following limits as a definite integrals. The number x_k^* is in the k-th subinterval of P and n is the number of intervals in P and Δx_k is the length of the k-th subinterval.

(a) $\displaystyle\lim_{||P||\to 0} \sum_{k=1}^{n} \frac{1}{1 + x_k^*}\Delta x_k$ where P is a partition of $[2, 6]$.

Solution. $\displaystyle\lim_{||P||\to 0} \sum_{k=1}^{n} \frac{1}{1 + x_k^*}\Delta x_k = \int_2^6 \frac{1}{1 + x}dx$

(b) $\displaystyle\lim_{||P||\to 0} \sum_{k=1}^{n} \frac{x_k^*}{4}\Delta x_k$ where P is a partition of $[1, 3]$

Solution. $\lim_{||P||\to 0} \sum_{k=1}^{n} \frac{x_k^*}{4} \Delta x_k = \int_1^3 \frac{x}{4}\,dx$

(c) $\lim_{||P||\to 0} \sum_{k=1}^{n} (x_k^*)^4 \Delta x_k$ where P is a partition of $[-2, 4]$

Solution. $\lim_{||P||\to 0} \sum_{k=1}^{n} (x_k^*)^4 \Delta x_k = \int_{-2}^4 x^4\,dx$

(d) $\lim_{||P||\to 0} \sum_{k=1}^{n} \sin^2(x_k^*) \Delta x_k$ where P is a partition of $[0, \pi]$

Solution. $\lim_{||P||\to 0} \sum_{k=1}^{n} \sin^2(x_k^*) \Delta x_k = \int_0^\pi \sin^2\,dx$

24. Find a formula for the Riemann sum for f obtained by dividing the given interval into n subintervals of equal length and using the right endpoint for each x_k^*. Then take the limit of the sum as $n \to \infty$.

(a) $f(x) = 4x$ on $[1, 3]$.

Solution. With $\Delta x_k = \frac{2}{n}$ and $x_k^* = 1 + \frac{2k}{n}$ we get the Riemann sum

$$\begin{aligned}\sum_{k=1}^{n} 4x_k^* \Delta x_k &= \sum_{k=1}^{n} 4\left(1 + \frac{2k}{n}\right)\frac{2}{n} = \frac{2}{n}\sum_{k=1}^{n} 4 + \frac{4\cdot 2}{n^2}\sum_{k=1}^{n} 2k \\ &= 8 + \frac{8}{n^2} 2\frac{n(n+1)}{2} = 8 + \frac{8(n+1)}{n}.\end{aligned}$$

Taking the limit as $n \to \infty$ we obtain

$$\lim_{n\to\infty} 8 + \frac{8(n+1)}{n} = 8 + 8 = 16.$$

(b) $f(x) = 6x^2$ on $[1, 3]$.

Solution. With $\Delta x_k = \frac{2}{n}$ and $x_k^* = 1 + \frac{2k}{n}$ we get the Riemann sum

$$\begin{aligned}\sum_{k=1}^{n} 6(x_k^*)^2 \Delta x_k &= \sum_{k=1}^{n} 6\left(1 + \frac{2k}{n}\right)^2 \frac{2}{n} = \frac{12}{n}\sum_{k=1}^{n}\left(1 + 2\frac{2k}{n} + \frac{4k^2}{n^2}\right) \\ &= \frac{12}{n}\left(n + 2\frac{2}{n}\frac{n(n+1)}{2} + \frac{4}{n^2}\frac{n(n+1)(2n+1)}{6}\right) \\ &= 12 + \frac{24(n+1)}{n} + \frac{8(n+1)(2n+1)}{n^2}.\end{aligned}$$

Taking the limit as $n \to \infty$ we obtain

$$\lim_{n\to\infty} 12 + \frac{24(n+1)}{n} + \frac{8(n+1)(2n+1)}{n^2} = 12 + 24 + 16 = 52.$$

25. Find a formula for the Riemann sum for f obtained by dividing the given interval into n subintervals of equal length and using the right endpoint for each x_k^*. Then take the limit of the sum as $n \to \infty$.

    (a) $f(x) = x$ on $[a, b]$ where $a \in \mathbb{R}$ and $b \in \mathbb{R}$ with $a < b$.

    Solution. Let $d = b - a$. With $\Delta x_k = \frac{d}{n}$ and $x_k^* = a + \frac{dk}{n}$ we get the Riemann sum

    $$\begin{aligned}\sum_{k=1}^{n} x_k^* \Delta x_k &= \sum_{k=1}^{n} \left(a + \frac{dk}{n}\right) \frac{d}{n} = \frac{d}{n} \sum_{k=1}^{n} a + \frac{d^2}{n^2} \sum_{k=1}^{n} k \\ &= da + \frac{d^2}{n^2} \frac{n(n+1)}{2} = da + \frac{d^2(n+1)}{2n}.\end{aligned}$$

    Taking the limit as $n \to \infty$ we obtain

    $$\lim_{n\to\infty} da + \frac{d^2(n+1)}{2n} = da + \frac{d^2}{2} = (b-a)a + \frac{(b-a)^2}{=} ba - a^2 + \frac{b^2 + a^2 - 2ba}{2} = \frac{b^2}{2} - \frac{a^2}{2}$$

    (b) $f(x) = x^2$ on $[a, b]$ where $a \in \mathbb{R}$ and $b \in \mathbb{R}$ with $a < b$.

    Solution. Let $d = b - a$. With $\Delta x_k = \frac{d}{n}$ and $x_k^* = a + \frac{dk}{n}$ we get the Riemann sum

    $$\begin{aligned}\sum_{k=1}^{n} (x_k^*)^2 \Delta x_k &= \sum_{k=1}^{n} \left(a + \frac{dk}{n}\right)^2 \frac{d}{n} = \frac{d}{n} \sum_{k=1}^{n} \left(a^2 + 2a\frac{dk}{n} + \frac{d^2k^2}{n^2}\right) \\ &= \frac{d}{n}\left(na^2 + \frac{2ad}{n} k \frac{n(n+1)}{2} + \frac{d^2}{n^2} \frac{n(n+1)(2n+1)}{6}\right) \\ &= a^2 d + \frac{ad^2(n+1)}{n} + \frac{d^3(n+1)(2n+1)}{6n^2}.\end{aligned}$$

    Taking the limit as $n \to \infty$ we obtain

    $$\begin{aligned}\lim_{n\to\infty} a^2 d + ad^2 \frac{n+1}{n} + d^3 \frac{(n+1)(2n+1)}{6n^2} &= a^2 d + ad^2 + \frac{1}{3} d^3 \\ &= a^2(b-a) + \frac{1}{2}a(b-a)^2 + \frac{1}{3}(b-a)^3 \\ &= a^2 b - a^3 + ab^2 - 2a^2 b + a^3 + \frac{1}{3}b^3 - ab^2 + a^2 b - \frac{1}{3}a^3 \\ &= \frac{1}{3}b^3 - \frac{1}{3}a^3\end{aligned}$$

Why are these limits equal to $\int_a^b f(x)\,dx$?

Solution. The definition of the definite integral says that f is integrable on $[a, b]$, if all limits of Riemann sums independent of the choice of the partitions or the values of the x_k^* are the same. By one of the Theorem from the lecture a function that is continuous on an interval $[a, b]$ is integrable on $[a, b]$. That means that all limits of Riemann sums independent of the choice of the partitions or the values of the x_k^* are the same if f is continuous. So, as the functions $f(x) = x$ and $f(x) = x^2$ are continous, it is sufficient to compute one limit of Riemann sums to find $\int_a^b f(x)\,dx$.

26. Find a formula for the Riemann sum for $f(x) = x^2 - 1$ obtained by dividing $[0, 3]$ into n subintervals of equal length and using right endpoints. Then take the limit of the sum as $n \to \infty$.

Solution. With $\Delta x_k = \frac{3}{n}$ and $x_k^* = \frac{3k}{n}$ we get the Riemann sum

$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k* = = \sum_{k=1}^{n} \left(\left(\frac{3k}{n}\right)^2 - 1 \right) \frac{3}{n} = \frac{27}{n^3} \sum_{k=1}^{n} k^2 - \frac{3}{n} \sum_{k=1}^{n} 1$$
$$= \frac{27}{n^3} \frac{n(n+1)(2n+1)}{6} - 3 = \frac{9(n+1)(2n+1)}{2n^2} - 3.$$

Taking the limit as $n \to \infty$ we obtain

$$\lim_{n\to\infty} \frac{9(n+1)(2n+1)}{2n^2} - 3 = 9 - 3 = 6$$

27. Interpret the following limit as the limit of a Riemann sum $\sum_{k=1}^{n} f(x_k^*)\Delta x_k$ of some function f with equally spaced sub-intervals of $[0, 1]$ using right endpoints:

$$L = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\left(\frac{k}{n}\right)^4 + 3 \right) \left(\frac{1}{n}\right).$$

Use the Fundamental Theorem of Calculus to find the exact value of the limit L. Fill in the blanks.

Δx_k = ____________

x_k^* = ____________

$f(x)$ = ____________

L = ____________

Solution. $\boxed{\begin{array}{ccc} \Delta x_k & = & 1/n \\ x_k^* & = & k/n \\ f(x) & = & x^4+3 \\ L & = & 16/5 \end{array}}$ To compute L, note that the limit of the Riemann sum is the definite integral $\int_0^1 x^4 + 3\,dx$. We compute that using FTC to get

$$\int_0^1 x^4 + 3\,dx = \left[\frac{x^5}{5} + 3x\right]_0^1 = \left(\frac{1}{5} + 3\right) - (0+0) = \frac{16}{5}.$$

28. Suppose that f and h are integrable and that

$$\int_1^7 f(x)\,dx = -1, \qquad \int_4^7 f(x)\,dx = 5, \qquad \int_4^7 g(x)\,dx = 6.$$

Find

(a) $\int_1^7 -2f(x)\,dx$

Solution. $\int_1^7 -2f(x)\,dx = (-2)\int_1^7 f(x)dx = (-2)(-1) = 2$

(b) $\int_4^7 f(x) + 3g(x)\,dx$

Solution. $\int_4^7 f(x) + 3g(x)\,dx = \int_4^7 f(x)\,dx + 3\int_4^7 g(x)\,dx = 5 + 3 \cdot 6 = 23$

(c) $\int_7^1 f(x)\,dx$

Solution. $\int_7^1 f(x)\,dx = -\int_1^7 f(x)dx = -(-1) = 1$

(d) $\int_1^4 f(x)\,dx$

Solution. $\int_1^4 f(x)\,dx = \int_1^7 f(x)\,dx - \int_4^7 f(x)\,dx = (-1) - 5 = -6$

29. Show that $\int_0^1 2\cos x\,dx$ cannot possibly be 3.

Solution. We have $\cos x \leq 1$ for all $x \in \mathbb{R}$. Therefore $\displaystyle\int_0^1 2\cos x\,dx = 2\int_0^1 \cos x\,dx < 2\int_0^1 1 = 2 < 3$.

30. Show that $0 \leq \displaystyle\int_0^5 (\sin x)^2\,dx \leq 5$.

Solution. We have $0 \leq (\sin x)^2$. Thus $0 = \int_0^5 0\,dx \leq \int_0^5 (\sin x)^2\,dx$. Also $(\sin x)^2 \leq 1$. Thus $\displaystyle\int_0^5 (\sin x)^2\,dx \leq \int_0^5 1\,dx = 5$.

31. State the mean value theorem for definite integrals.

Solution. If f is continuous on $[a,b]$, then there exists $c \in (a,b)$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

32. State the fundamental theorem of calculus part I.

Solution. If f is continuous on $[a,b]$, then the function

$$F(x) = \int_a^x f(t)\,dt$$

is continuous on $[a,b]$ and differentiable on (a,b), and its derivative is

$$F'(x) = f(x).$$

33. State the fundamental theorem of calculus part II.

Solution. If f is continuous on $[a,b]$ and F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

34. Evaluate the integrals

(a) $\displaystyle\int_{-2}^{2} x^3 - 2x + 3\,dx$

Solution.

$$\begin{aligned}\int_{-2}^{2} x^3 - 2x + 3\,dx &= \left[\frac{1}{4}x^4 - x^2 + 3x\right]_{-2}^{2} \\ &= \frac{1}{4}2^4 - 2^2 + 3\cdot 2 - \left(\frac{1}{4}(-2)^4 - (-2)^2 + 3(-2)\right) \\ &= 4 - 4 + 6 - 4 + 4 + 6 = 12\end{aligned}$$

(b) $\displaystyle\int_{0}^{\pi} 1 + \cos x\,dx$

Solution. $\displaystyle\int_{0}^{\pi} 1 + \cos x\,dx = [x + \sin x]_0^{\pi} = \pi + \sin\pi - (0 + \sin 0) = \pi$

35. Find the average value of $f(x) = -5x^2 - 1$ on the interval $[0, 2]$.

Solution. The average of f on $[0, 2]$ is

$$\frac{1}{2-0}\int_0^2 -5x^2 - 1 dx = \frac{1}{2}\left[-5\frac{x^3}{3} - x\right]_0^2 == \frac{1}{2}\left(\left(-5\frac{8}{3} - 2\right) - 0\right) = \frac{1}{2}\left(\frac{-40}{3} - 2\right) = \frac{-23}{3}.$$

36. Find the area between the curve $y = (x-2)(x-4)(x-6)$ on $[-1, 7]$ and the x-axis.

Solution. The curve intersects the x-axis at $x = 2$, $x = 4$, and $x = 6$.
An antiderivative of

$$f(x) = (x-2)(x-4)(x-6) = x^3 - 12x^2 + 44x - 48$$

is

$$F(x) = \frac{1}{4}x^4 - 4x^3 + 22x^2 - 48x.$$

The area between the curve and the x-axis is

$$\begin{aligned}&-\int_{-1}^{2} f(x)\,dx + \int_{2}^{4} f(x)\,dx - \int_{4}^{6} f(x)\,dx + \int_{6}^{7} f(x)\,dx \\ &= -[F(x)]_{-1}^{2} + [F(x)]_2^4 - [F(x)]_4^6 + [F(x)]_6^7 \\ &= -\frac{-441}{4} + 4 - (-4) + \frac{25}{4} = \frac{249}{2} = 124.5\end{aligned}$$

37. Find the area of the region between the curve $y = 3 - x^2$ and the line $y = -1$.

Solution. The line and the curve intersect in $(-2, -1)$ and $(2, -1)$. The area is

$$\int_{-2}^{2} 3 - x^2 - (-1)\,dx = \int_{-2}^{2} 4 - x^2\,dx = \left[4x - \frac{1}{3}x^3\right]_{-2}^{2} = 8 - \frac{8}{3} - \left(-8 + \frac{8}{3}\right) = \frac{32}{3}$$

38. Let $f(x) = 1 - x^2$ and $g(x) = x - 1$. Find the area between the curves $y = f(x)$ and $y = g(x)$ between $x = -1$ and $x = 2$.

Solution. The two functions intersect when $1 - x^2 = x - 1$ or equivalently $2 = x + x^2$, that is when $x = 1$ or $x = -2$. As $1 = f(0) > g(0) = -1$ and $-\frac{5}{4}f(\frac{3}{2}) < g(\frac{1}{2}) = \frac{1}{2}$, the area between the curves is

$$A = \int_{-1}^{1} f(x) - g(x)\,dx + \int_{1}^{2} g(x) - f(x).$$

We get

$$\begin{aligned}\int_{-1}^{1} f(x) - g(x)\,dx &= \int_{-1}^{1} 1 - x^2 - (x-1)\,dx = \int_{-1}^{1} 2 - x^2 - x\,dx \\ &= \left[2x - \frac{1}{3}x^3 - \frac{1}{2}x^2\right]_{-1}^{1} = 2 - \frac{1}{3} - \frac{1}{2} - (-2 + \frac{1}{3} - \frac{1}{2}) = 4 - \frac{2}{3} = \frac{10}{3}\end{aligned}$$

and

$$\begin{aligned}\int_{1}^{2} g(x) - f(x)\,dx &= \int_{1}^{2} x - 1 - (1 - x^2)\,dx = \int_{1}^{2} x - 2 + x^2\,dx \\ &= \left[\frac{1}{2}x^2 - 2x + \frac{1}{3}x^3\right]_{1}^{2} = 2 - 4 + \frac{8}{3} - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)\frac{7}{3} - 12 = \frac{11}{6}\end{aligned}$$

Thus the area between the curves is $A = \frac{10}{3} + \frac{11}{6} = \frac{31}{6}$.