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The Euclidean Algorithm for the integers is well known and yields a finite continued fraction expansion for each rational number. Geometrically, successive convergents in this expansion correspond to endpoints of edges in the Farey tessellation of the complex upper half plane. The sequence of convergents thus describes a path from the point at infinity to a given rational number, following edges of the tessellation; by identifying points in the upper half plane with positive definite binary quadratic forms (up to scaling), we express this path as a product of matrices in $\mathrm{SL}_2(\mathbb{Z})$. In general, when the ring of integers of a quadratic number field is Euclidean, there exists a suitable Euclidean function and algorithm with which we may construct a continued fraction expansion for each field element. In these cases, we prove the analogous result that pairs of adjacent convergents determine edges in the Voronoi tessellation of hyperbolic 3-space. We identify points in the upper half space with positive definite binary Hermitian forms (up to scaling), and express the resulting path as a product of matrices in $\mathrm{GL}_2(\mathcal{O}_F)$. Finally, since the Voronoi tessellation exists for all imaginary quadratic fields, including those with non-Euclidean rings of integers, we explore the extent to which this geometric interpretation of continued fractions holds in the general case.

A GEOMETRIC GENERALIZATION OF CONTINUED FRACTIONS FOR IMAGINARY QUADRATIC FIELDS

by

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Introduction

1.1 History and Motivation

The mathematics behind continued fractions has been studied, in one form or another, for millennia. Euclid's algorithm for the greatest common divisor of two integers, developed around the third century B.C.E. [Bre91], constructs a finite continued fraction, though it was not presented in those terms. While the modern version of this well-known algorithm bears Euclid's name, the construction of sequences of partial quotients from ratios of line segments was studied by the Greeks in the centuries before Euclid, and similar work from much earlier periods has been found. The first known written reference to a rule with a continued fractions flavor appears on a Babylonian tablet from around 2000 B.C.E. [Bre91]. Early attempts to measure circles and find approximations to the value of π use similar methods, and ratios now known to be the first few continued fraction convergents to π appear in the work of early Indian and Chinese mathematicians.

Some of the familiar modern notation for continued fractions appears in attempts to solve Diophantine equations, notably by the Indian mathematician Aryabhata (c. 500 A.D.) and subsequent interpretations of his work [Bre91]. In particular, Aryabhata knew of the recursive formula for computing continued fraction convergents as well as the relation between successive convergents $\frac{p_{n-1}}{q_{n-1}}$ and $\frac{p_n}{q_n}$, namely

$$p_n q_{n-1} - q_n p_{n-1} = (-1)^n.$$

Brahmagupta improved on Aryabhata's work and used it to solve first- and second-degree Diophantine equations arising from astronomy [Bre91]. The Persian mathematician Omar Khayyam's book on algebra from around 1100 A.D. includes a method for determining whether two ratios are equal, involving a recursive sequence of integers that amounts to comparing the continued fraction expansions of the given ratios [Bre91]. It was not until 1655 that the first use of modern notation for continued fractions appeared in Europe. In the modern era, continued fractions are involved in the study of many aspects of number theory, Pell's equation, transcendental numbers, probability theory, physics, and many other areas.

The known relationship between adjacent continued fraction convergents allows for their representation as products of 2×2 matrices with integer entries and determinant ± 1. The matrix group $\mathrm{SL}_2(\mathbb{Z})$ acts on the complex upper half plane $\mathcal{H}$, and continued fractions may be represented by paths of connected edges in the Farey tessellation of $\mathcal{H}$. The sequences of matrices resulting from this construction are used in explicit computations with classical modular forms.

Analogues of both the algebraic and geometric constructions of rational continued fractions exist for the Gaussian integers [Hoc19], and can be more or less immediately generalized to imaginary quadratic fields with Euclidean rings of integers. For general imaginary quadratic fields, we still have a tessellation of hyperbolic 3-space $\mathbb{H}^3$, but in general do not have a Euclidean ring of integers to allow for the algebraic construction and extraction of the matrix convergents. Cremona [Cre84] and several his students ([Ara10, Whi90, Lin05, Byg98], and others) explicitly computed tessellations of hyperbolic 3-space for selected imaginary quadratic number fields F, constructing a fundamental region for the action of $\mathrm{GL}_2(F)$, identifying cusps in $\mathbb{P}^1(F)$ with vertices of hyperbolic polyhedra, and relating edges of the polyhedra to modular symbols. The extracted information is used for homology computations. This ground-up approach must be adjusted to account for the differences in the geometry of each field, which becomes more challenging as the class number $h(F)$ increases.

An alternative approach – one which is employed in this work – is to start with the Voronoi tessellation of hyperbolic 3-space. The connection between the Voronoi tessellation of the complex upper half plane and classical continued fractions is well studied, and an analogous tessellation of $\mathbb{H}^3$ exists and is readily computed for every imaginary quadratic field. Equivalence classes of edges in the Voronoi tessellation provide the necessary information to modify the continued fraction algorithm for all imaginary quadratic fields of class number 1, and using this new approach we obtain the updated algorithm presented in Section 3.1. Future work in this area could focus on using the full Voronoi polytope data to overcome the obstacle posed by singular points in fields of class number greater than 1.

1.2 Classical Continued Fractions

Definition 1.1. A *finite continued fraction* is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_n}}}},$$

with *elements* $a_i, i = 0, 1, \ldots, n$, $a_i \in \mathbb{Z}$, $a_i \geq 1$ for $i \geq 1$.

Note that there exist other possible definitions of continued fractions, for example where the numerators are not necessarily all equal to 1, or where the conditions on the a_i are modified.

Since all of the elements are rational numbers, and the finite continued fraction is the result of finitely many rational operations on its elements, the expression is equal to a rational number. Every rational number has a (not necessarily unique) representation as a finite continued fraction [Khi64]. We will choose to represent a rational number α by a finite continued fraction in which all of the elements are nonzero and $a_n \neq 1$. This allows the convenient and more compact notation

$$\alpha = [a_0; a_1, a_2, \ldots, a_n].$$

By truncating this representation at the element a_k, we obtain the k^{th}-*order convergent* to α, the rational number $\dfrac{p_k}{q_k}$. Note that α itself is the n^{th}-order convergent, $\alpha = \dfrac{p_n}{q_n}$. The value of any convergent may be computed by simplifying the continued fraction, since finitely many operations are needed; however, there is a recursive formula for computing the k^{th}-order convergent directly:

Theorem 1.2 ([Khi64, Theorem 1])**.** *Set $p_{-1} = 1$, $q_{-1} = 0$ $p_0 = a_0$, and $q_0 = 1$. For $k \geq 1$,*

$$\begin{cases} p_k = a_k p_{k-1} + p_{k-2} \\ q_k = a_k q_{k-1} + q_{k-2}. \end{cases}$$

For all pairs of successive convergents, the following property holds:

Theorem 1.3 ([Khi64, Theorem 2])**.** *For all $k \geq 0$,*

$$q_k p_{k-1} - p_k q_{k-1} = (-1)^k.$$

This naturally suggests a redefinition of the recurrence relation in terms of square integer matrices having determinant ± 1. By setting $\begin{bmatrix} p_{-1} & p_0 \\ q_{-1} & q_0 \end{bmatrix} = \begin{bmatrix} 1 & a_0 \\ 0 & 1 \end{bmatrix}$, we have

$$\begin{bmatrix} p_k & p_{k+1} \\ q_k & q_{k+1} \end{bmatrix} = \begin{bmatrix} p_{k-1} & p_k \\ q_{k-1} & q_k \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & a_{k+1} \end{bmatrix}$$

for all $k \in \mathbb{Z}_{\geq 0}$ [Sch09a].

When we truncate the continued fraction representation to obtain a convergent to α, we may also consider the *remainder* of the continued fraction,

$$r_k = [a_k; a_{k+1}, a_{k+2}, \ldots, a_n].$$

Then we may write

$$[a_0; a_1, a_2, \ldots, a_n] = [a_0; a_1, a_2, \ldots, a_{k-1}, r_k],$$

and by means of the recursive formula above, we have the following result:

Theorem 1.4 ([Khi64, Theorem 5])**.** *For arbitrary k $(1 \leq k \leq n)$,*

$$[a_0; a_1, a_2, \ldots, a_n] = \frac{p_{k-1} r_k + p_{k-2}}{q_{k-1} r_k + q_{k-2}}.$$

The convergents $\dfrac{p_k}{q_k}$ are best (Diophantine) approximations of the first kind, i.e., they minimize $\left|\frac{p}{q} - \alpha\right|$ for all $p, q \in \mathbb{Z}$ with $1 \leq q \leq q_k$ [Sch09a].

In this work, we are primarily interested in the sequence of convergents rather than the finite continued fraction expression itself, and will compute these convergents directly.

1.3 Euclidean and Pseudo-Euclidean Functions

The construction of continued fractions for elements in the field of rational numbers $\mathbb{Q}$ relies on the fact that its ring of integers $\mathbb{Z}$ is a Euclidean ring.

Definition 1.5. A *Euclidean function* on a ring R is any function $N : R \to \mathbb{Z}_{\geq 0}$ satisfying the condition that for all $\alpha, \beta \in R$ with $\beta \neq 0$, there exist $q, r \in R$ such that $\alpha = q\beta + r$ and $N(r) < N(\beta)$. A ring on which a Euclidean function can be defined is said to be a *Euclidean ring*.

For example, the absolute value function is a Euclidean function on the ordinary integers $\mathbb{Z}$. To investigate continued fractions over imaginary quadratic number fields, it is useful to know when the ring of algebraic integers is Euclidean. First we recall some definitions and key facts about number fields from [Neu92].

Definition 1.6. An *algebraic number field* is a finite-degree field extension K of $\mathbb{Q}$. The degree of an extension field K over $\mathbb{Q}$ is denoted by $[K:\mathbb{Q}]$. If $[K:\mathbb{Q}]=2$, then K is called a *quadratic number field.*

Definition 1.7. The elements of an algebraic number field K are called *algebraic numbers*, or *quadratic numbers* in the case $[K:\mathbb{Q}]=2$. If an algebraic number occurs as the zero of a monic polynomial in $\mathbb{Z}[x]$, it is called an *algebraic integer*. The set of algebraic integers of K forms an integral domain, denoted by $\mathcal{O}_K$.

Note that every nonzero, non-unit element in $\mathcal{O}_K$ can be factored into a product of irreducible elements, but this factorization is not necessarily unique. We do, however, have unique factorization of the nonzero ideals of $\mathcal{O}_K$ into products of powers of prime ideals (up to ordering). This allows us to discuss the notion of a class group, which can be thought of as a measure of how far the ring of integers is from being a principal ideal domain; see Section 2.5 for details.

Letting R be the ring of algebraic integers $\mathcal{O}_F$ of an imaginary quadratic number field F, and defining N to be the square of the complex norm, $N(\alpha)=|\alpha|^2$, the conditions in Definition 1.5 may be rewritten as

$$\frac{\alpha}{\beta}=q+\frac{r}{\beta} \quad \text{and} \quad \left|\frac{r}{\beta}\right|<1.$$

Thus to characterize the ring of integers $\mathcal{O}_F$ as Euclidean, it suffices to find, for each pair $\alpha,\beta\in\mathcal{O}_F$ with $\beta\neq 0$, an element $q\in\mathcal{O}_F$ such that $\left|\frac{\alpha}{\beta}-q\right|<1$. It is known that there are exactly five imaginary quadratic fields with Euclidean rings of integers, namely $\mathbb{Q}(\sqrt{-d})$ for $d=1,2,3,7,11$ [Neu92].

The existence of a Euclidean function on a ring gives rise to a function on the field of fractions, satisfying a similar property. To accommodate the situation when $r=0$, we define a function on $F\cup\{\infty\}$:

Proposition 1.8. *Let R be a Euclidean ring, and let F be the field of fractions of R. Then there exists a function $\eta: F\cup\{\infty\}\to\mathbb{Z}_{\geq 0}$ with the property that for every $\frac{\alpha}{\beta}\in F$, there exists $q\in R$ such that $\eta\left(\frac{1}{\frac{\alpha}{\beta}-q}\right)<\eta\left(\frac{\alpha}{\beta}\right)$.*

Proof. Let R be a Euclidean ring with Euclidean function N, and let F be the field of fractions of R. Let $\alpha',\beta'\in R$ with $\beta'\neq 0$, and let $\frac{\alpha}{\beta}$ be the reduced representative of the class of $\frac{\alpha'}{\beta'}$ in F, i.e., $\alpha, \beta \in R$ with $\beta \neq 0$, α and β share no common factors, and $\frac{\alpha'}{\beta'} = \frac{\alpha}{\beta}$.

By the definition of Euclidean function, there exist $q, r \in R$ such that $\alpha = q\beta + r$ and $N(r) < N(\beta)$. Then $\alpha - q\beta = r$, and since $\beta \neq 0$, by arithmetic in F we have $\frac{\alpha}{\beta} - q = \frac{r}{\beta}$.

Define $\eta\left(\frac{\alpha}{\beta}\right) = N(\beta)$ for any reduced representative $\frac{\alpha}{\beta} \in F$, and set $\eta(\infty) = 0$. If $r = 0$, then $\frac{\alpha}{\beta} \in R$, and $\eta\left(\frac{1}{\alpha/\beta - q}\right) = \eta\left(\frac{1}{0}\right) = 0 < \eta\left(\frac{\alpha}{\beta}\right)$. Otherwise,

$$
\begin{aligned}
\eta\left(\frac{1}{\alpha/\beta - q}\right) &= \eta\left(\frac{1}{r/\beta}\right) \\
&= \eta\left(\frac{\beta}{r}\right) \\
&= N(r) \\
&< N(\beta) \\
&= \eta\left(\frac{\alpha}{\beta}\right).
\end{aligned}
$$

□

In the theory of hemispheres presented by [Swa71], reduced ratios of algebraic integers are identified with cusps in $\mathbb{P}^1(F)$. The radius of the hemisphere determined by a principal cusp $\frac{\lambda}{\mu}$ is controlled by the norm of the denominator. In particular, with F embedded in $\mathbb{C}$, the radius of the hemisphere over each integral cusp has complex norm 1. Thus for imaginary quadratic fields where the ring of integers is Euclidean, we have that every point in F lies under an integral hemisphere. When the ring of integers is not Euclidean, there exist points in F lying outside every integral hemisphere.

For those cases where it is not possible to characterize the ring of integers as Euclidean by the above inequality, the following, more general condition holds:

Proposition 1.9 ([Swa71, Proposition 3.11])**.** *Let $F = \mathbb{Q}(\sqrt{-d})$ be a quadratic imaginary field with ring of integers $\mathcal{O}_F$. If $\frac{\alpha}{\beta} \in F$, we can find $\mu, \lambda \in \mathcal{O}_F$ such that $\langle \mu, \lambda \rangle = \mathcal{O}_F$ and $\left|\frac{\mu\alpha}{\beta} - \lambda\right| \leq 1$. The points for which we cannot find μ, λ such that the above inequality is strict are called* singular points*; these points all lie in F, and there are finitely many up to translation by elements of $\mathcal{O}_F$.*

To extend Proposition 1.8 and the function η to fields whose rings of integers are not Euclidean, we generalize the notion of inversion, then define an analogous "pseudo-Euclidean" function on the field.

Definition 1.10. Let F be a field. The *projective line over* F, denoted by $\mathbb{P}^1(F)$, is the set of equivalence classes

$$\mathbb{P}^1(F) = \left\{(\alpha, \beta) \in F^2 \setminus \{(0,0)\}\right\} / \sim ,$$

where $(\alpha, \beta) \sim (x\alpha, x\beta)$ for $x \in F \setminus \{0\}$.

We identify $\mathbb{P}^1(F)$ with $F \cup \{\infty\}$ by $(\alpha, \beta) \mapsto \frac{\alpha}{\beta}$ if $\beta \neq 0$, and $(\alpha, \beta) \mapsto \infty$ if $\beta = 0$.

Definition 1.11. Let F be a field. An *inversion* is a bijection on the set of nonzero points of F. If F is an imaginary quadratic field, we extend this definition to include 0 and ∞; an inversion is a bijection on the set of cusps $\mathbb{P}^1(F)$.

Definition 1.12. Let R be an integral domain, and F the fraction field of R. A function $\psi : F \cup \{\infty\} \to \mathbb{Z}_{\geq 0}$ is called a *pseudo-Euclidean function* if there exists a finite set S of representatives in the quotient of F by R, together with a finite set of inversions U, such that for every $z \in F \setminus S$ there exists $q \in R$ and $u \in U$ such that $\psi(u(z - q)) < \psi(z)$.

The finite set S for the above definition contains the singular points referenced in Proposition 1.9. This becomes important when we look at cases with class number greater than 1; when the class group is trivial, the set S is empty.

The function η defined in Theorem 1.8 on the field of fractions of a Euclidean ring is a special case of a pseudo-Euclidean function, where we take η to be the norm of the denominator, the set of singular points is empty, and the set of inversions consists of a single element: the usual $z \mapsto \frac{1}{z}$.

1.4 Euclidean and Pseudo-Euclidean Algorithms

With our pseudo-Euclidean function, we will construct a pseudo-Euclidean algorithm. First we recall the usual Euclidean Algorithm:

Theorem 1.13 (Euclidean Algorithm)**.** *Let R be a Euclidean ring with Euclidean function N, and let $a, b \in R$ with $b \neq 0$. By repeated application of the Euclidean function, we find elements $q_i, r_i \in R$ such that*

$$\begin{aligned}
&a = q_1 b + r_1, && N(r_1) < N(b),\\
&b = q_2 r_1 + r_2, && N(r_2) < N(r_1),\\
&r_1 = q_3 r_2 + r_3, && N(r_3) < N(r_2),\\
&\vdots && \vdots\\
&r_{n-2} = q_n r_{n-1} + r_n, && N(r_n) < N(r_{n-1}),\\
&r_{n-1} = q_{n+1} r_n + 0, && 0 < N(r_n).
\end{aligned}$$

The element r_n (the last nonzero remainder in the division process) is the greatest common divisor of a and b.

Each inequality holds by the definition of Euclidean function, and since the output of the Euclidean function N is a non-negative integer, the process terminates in finitely many steps.

The Euclidean algorithm is used to construct a rational continued fraction of the form given in Definition 1.1, where $\alpha = \frac{a}{b}$ with $a, b \in \mathbb{Z}$, $b \neq 0$, the Euclidean function on $\mathbb{Z}$ is the absolute value function, the elements a_i are the computed quotients q_i, and inversion $z \mapsto \frac{1}{z}$ is applied to the successive remainders r_i.

Working instead in the field of fractions F of R, and using the related function η from Theorem 1.8, we can reinterpret the Euclidean algorithm to emphasize that the mechanics of computing a continued fraction are an alternating sequence of translations and inversions.

Theorem 1.14 (Euclidean Algorithm in Fractions)**.** *Let F be the field of fractions of a Euclidean ring R, and let $\frac{\alpha}{\beta}$ in F. By repeated application of the function η defined in the proof of Theorem 1.8, we can find elements $q_i, r_i \in R$ such that*

$$\begin{array}{lll}
\dfrac{\alpha}{\beta} = q_1 + \dfrac{r_1}{\beta} & \Rightarrow \quad \dfrac{\alpha}{\beta} - q_1 = \dfrac{r_1}{\beta}, & \eta\left(\dfrac{1}{{}^{r_1}/_{\beta}}\right) < \eta\left(\dfrac{\alpha}{\beta}\right), \\
\dfrac{\beta}{r_1} = q_2 + \dfrac{r_2}{r_1} & \Rightarrow \quad \dfrac{\beta}{r_1} - q_2 = \dfrac{r_2}{r_1}, & \eta\left(\dfrac{1}{{}^{r_2}/_{r_1}}\right) < \eta\left(\dfrac{\beta}{r_1}\right), \\
\vdots & \qquad \vdots & \vdots \\
\dfrac{r_{n-2}}{r_{n-1}} = q_n + \dfrac{r_n}{r_{n-1}} & \Rightarrow \quad \dfrac{r_{n-2}}{r_{n-1}} - q_n = \dfrac{r_n}{r_{n-1}}, & \eta\left(\dfrac{1}{{}^{r_n}/_{r_{n-1}}}\right) < \eta\left(\dfrac{r_{n-2}}{r_{n-1}}\right), \\
\dfrac{r_{n-1}}{r_n} = q_{n+1} + 0 & \Rightarrow \quad \dfrac{r_{n-1}}{r_n} - q_{n+1} = 0, & 0 < \eta\left(\dfrac{r_{n-1}}{r_n}\right).
\end{array}$$

Each inequality on the right-hand side holds by Theorem 1.8, and termination in finitely many steps is guaranteed since η takes non-negative integer values.

In general, the ring of integers of an imaginary quadratic field is not Euclidean, so given an arbitrary $\frac{\alpha}{\beta} \in F$, we are not guaranteed to be able to find $q \in R$ such that $\left|\frac{\alpha}{\beta} - q\right| < 1$. In this case, the inversion $z \mapsto \frac{1}{z}$ is not enough to force the remainders to decrease as required for the algorithm to terminate. If, however, we can find a pseudo-Euclidean function on $\mathbb{P}^1(F)$ satisfying Definition 1.12, we can generalize Theorem 1.14 to this case.

Theorem 1.15 (Pseudo-Euclidean Algorithm)**.** *Let ψ be a pseudo-Euclidean function on the field of fractions F of an integral domain R, and let S and U be the finite sets given by Definition 1.12. Let $z \in \mathbb{P}^1(F)$.*

If $z \in F \setminus S$, by repeated application of ψ we can find $q_i \in R$ and $r_i \in \mathbb{P}^1(F)$ such that

$$\begin{aligned}
&z - q_1 = r_1, && \psi(u_1(r_1)) < \psi(z) \\
&u_1(r_1) - q_2 = r_2, && \psi(u_2(r_2)) < \psi(u_1(r_1)) \\
&\vdots && \vdots \\
&u_{n-1}(r_{n-1}) - q_n = r_n, && \psi(u_n(r_n)) < \psi(u_{n-1}(r_{n-1})) \\
&u_n(r_n) - q_{n+1} = r_{n+1}, && 0 < \psi(u_n(r_n)),
\end{aligned}$$

where $u_1, u_2, \ldots, u_n \in U$ are inversions (not necessarily distinct), and $u_n(r_n) \in S \cup \{\infty\}$.

To apply this theorem to imaginary quadratic fields in which the ring of algebraic integers $\mathcal{O}_F$ is a principal ideal domain, i.e. F has class number $h(F) = 1$, we will compute a finite set of inversions in $\mathrm{GL}_2(\mathcal{O}_F)$ using Voronoi tessellations of hyperbolic 3-space.

Chapter 2

Voronoi Tessellations

2.1 Binary Quadratic Forms and the Upper Half Plane

Here we summarize some definitions and key facts about quadratic forms from [BV07, Leh19, Sch09a].

Definition 2.1. A *binary quadratic form* over the set of real numbers $\mathbb{R}$ is a homogeneous polynomial of degree two in two variables:

$$f(x, y) = ax^2 + bxy + cy^2;\ \ a, b, c \in \mathbb{R},$$

with *discriminant* $\Delta(f) = b^2 - 4ac$. If the discriminant of f is positive, we say that f is *indefinite*; if $\Delta(f)$ is negative and a is positive, we say that f is *positive definite*; if $\Delta(f)$ is negative and a is negative, we say that f is *negative definite*.

To each quadratic form f we can associate a 2×2 symmetric matrix M, which allows us to evaluate the form on integers x and y via matrix multiplication.

Definition 2.2. For a binary quadratic form $f(x, y) = ax^2 + bxy + cy^2$, the *matrix of* f is the 2×2 symmetric matrix M_f with entries in $\mathbb{R}$ given by

$$M_f = \begin{bmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{bmatrix},$$

so that $f(x, y) = \begin{bmatrix} x & y \end{bmatrix} M_f \begin{bmatrix} x \\ y \end{bmatrix}$.

The value a quadratic form f takes on an ordered pair of integers (x_0, y_0) is equal to the result of $v^t M_f v$, where $v = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$. Note that if f is positive definite, then $f(x, y) > 0$ for all ordered pairs of integers (x, y) other than $(0, 0)$.

Definition 2.3. The arithmetic *minimum* of a quadratic form f is the minimum value of $f(x, y)$ for all $(x, y) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$.

Definition 2.4. A *minimal vector* of a quadratic form f is a vector $(x_0, y_0) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ such that $f(x_0, y_0)$ is the arithmetic minimum of f.

Let $\Gamma = \mathrm{SL}_2(\mathbb{R})$ be the group of matrices with determinant 1,

$$\Gamma = \left\{ g = \begin{bmatrix} q & r \\ s & t \end{bmatrix} : q, r, s, t \in \mathbb{R}; qt - rs = 1 \right\}.$$

The group Γ acts on the 3-dimensional real vector space V of binary quadratic forms by $g \cdot A = gAg^t$, where $g \in \Gamma$ and $A \in V$. Define an equivalence relation $\sim$ on the set of quadratic forms of discriminant Δ by $f \sim f'$ if and only if $M'_f = gM_f g^t$ for some $g \in \Gamma$. In other words, two forms are equivalent if and only if their orbits under the action of Γ are the same. When evaluated on the same vectors, equivalent forms produce the same values. In particular, two equivalent forms have the same minimum.

Note that scaling a quadratic form changes the value of its arithmetic minimum but does not change the set of minimal vectors; for computational convenience, we will often scale forms to a minimal value of 1.

A quadratic form is called *perfect* if it is completely determined by its arithmetic minimum and set of minimal vectors. Up to arithmetical equivalence and scaling by positive real numbers, there are finitely many perfect forms [Sch09b].

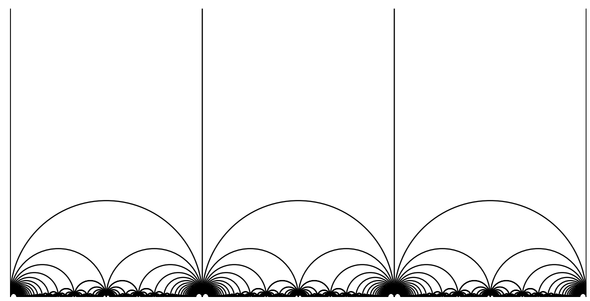
Identifying the minimal vectors of a perfect form with points in $\mathbb{Q} \cup \infty$, we have an ideal triangle in the complex upper half plane $\mathcal{H}$ corresponding to each perfect form. We have an action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathcal{H}$; in fact, we may use the action of $\mathrm{SL}_2(\mathbb{Z})$, which stabilizes $\mathbb{Q} \cup \infty$, to obtain a tessellation of $\mathcal{H}$ by triangles associated to perfect forms.

Definition 2.5. The *upper half plane*, denoted by $\mathcal{H}$, is a model of two-dimensional hyperbolic geometry consisting of the set of complex numbers with positive imaginary part,

$$\mathcal{H} = \{s + it : s, t \in \mathbb{R}, t > 0\}.$$

There is a correspondence between points in the upper half plane and positive definite binary quadratic forms. The group $\Gamma = \mathrm{SL}_2(\mathbb{R})$ acts transitively on $\mathcal{H}$ by fractional linear transformations, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \dfrac{az + b}{cz + d}$, thus we may generate $\mathcal{H}$ by the point

Figure 2.1. The Farey Tessellation of $\mathcal{H}$. Each ideal triangle in the tessellation corresponds to a perfect quadratic form.



$0 + 1i$ under this action. The stabilizer of the point i is the set

$$\begin{aligned}\mathrm{stab}_{\mathrm{SL}_2(\mathbb{R})}(i) &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R}) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot i = i \right\} \\ &= \left\{ A \in \mathrm{SL}_2(\mathbb{R}) : AA^t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\},\end{aligned}$$

thus $\mathrm{stab}_{\mathrm{SL}_2(\mathbb{R})}(i) \simeq \mathrm{SO}(2)$. Then we identify $\mathcal{H}$ with $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$.

Letting the identity quadratic form $x^2 + y^2$ be the representative whose stabilizer is $\mathrm{SO}(2)$, we identify points in the upper half plane with the cone of positive definite quadratic forms (mod positive scaling) via the map

$$s + it \mapsto \begin{bmatrix} 1 & -s \\ -s & s^2 + t^2 \end{bmatrix}.$$

Then the point $i = 0 + 1i \in \mathbb{C}$ is identified with the form $x^2 + y^2$, with matrix representation $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

The *Farey tessellation* of $\mathcal{H}$, the orbit of the positive imaginary axis under $\mathrm{SL}_2(\mathbb{Z})$, is shown in Figure 2.1. This tessellation is equivalent to the Voronoi tessellation by ideal triangles corresponding to perfect forms.

In the theory of perfect quadratic forms over $\mathbb{Q}$ due to Voronoi [Vor08], it is shown that there exists an infinite polyhedron in the space of quadratic forms on which the arithmetic group $\Gamma = \mathrm{GL}_n(\mathbb{Z})$ acts. The faces of this polyhedron determine the possible configurations of minimal vectors of quadratic forms, and in the case of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, there is exactly one perfect form up to arithmetical equivalence and scaling. In this case, the Voronoi polyhedron descends modulo scaling to the Farey tessellation of $\mathcal{H}$ by the ideal triangle with vertices $\{0, 1, \infty\}$.

2.2 Hemispheres and Quadratic Forms

With the identification between points in the upper half plane and positive definite binary quadratic forms, the theory of hemispheres over rational points can now be described in terms quadratic forms. For a form Q with matrix M_Q, denote evaluation of Q on a vector v by $M_Q[v]$. The set of forms evaluating the same on 0 and ∞, i.e., the set of all Q for which

$$M_Q \begin{bmatrix} 1 \\ 0 \end{bmatrix} = M_Q \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

is a one-parameter family of forms defining a semicircle in the upper half plane. This semicircle, centered at 0 with radius 1, is the hemisphere determined by the rational point 0/1.

Let $v' = [p'\ q']^t$ be a vector in $\mathbb{Z}^2$ such that p'/q' is not an integer. Up to translation by integers, v' is equivalent to a vector $v = [p\ q]^t$ with $|p| < |q|$, $\gcd(p, q) = 1$, and $0 < q \neq 1$. Then $-1 < p/q < 1$, so p/q lies under the hemisphere determined by $0/1$. It can be shown that the entire hemisphere determined by p/q lies under this integer hemisphere.

In general, the hemisphere determined by p/q consists of quadratic forms which give the same result upon evaluation at $[1\ 0]^t$ and $[p\ q]^t$. Evaluating a generic form Q at $v = [1\ 0]^t$, we find that

$$Q(v) = v^t \begin{bmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{bmatrix} v = a.$$

Since the identification between points in the upper half plane and positive definite binary quadratic forms holds up to scaling by positive real numbers, for convenience we scale the output to 1. For a fixed p/q, the set of forms Q such that $Q(1, 0) = Q(p, q)$ defines a hemisphere centered at p/q of radius $1/|q|$. Note that in $\mathcal{H}$, a "hemisphere" is a semicircle whose center lies on the boundary.

It can be shown that if the point in $\mathcal{H}$ corresponding to a given quadratic form Q lies on a hemisphere over a rational point p/q, but lies outside all other rational hemispheres, then the set of minimal vectors of Q is exactly $\{p/q, \infty\}$.

Lemma 2.6. *If a point $z \in \mathcal{H}$ lies on the hemisphere determined by the rational point $\frac{p}{q}$, but does not lie on or under the hemisphere determined by any $\frac{p'}{q'}$ with $\frac{p'}{q'} \neq \frac{p}{q}$, then the set $\left\{\infty, \frac{p}{q}\right\}$ represents an edge in the Voronoi tessellation of $\mathcal{H}$.*

Proof. Let $z = s + it \in \mathcal{H}$, and assume z lies on the hemisphere determined by the cusp $\frac{p}{q}$, but z does not lie on or under any hemisphere determined by $\frac{p'}{q'}$ with $\frac{p'}{q'} \neq \frac{p}{q}$. Then the quadratic form Q associated to z evaluates the same on ∞ and $\frac{p}{q}$. Since z

does not lie on or under the hemisphere determined by $\frac{p'}{q'}$, Q evaluates lower on ∞ than $\frac{p'}{q'}$. Thus

$$Q\left(\frac{p}{q}\right) = Q(\infty) < Q\left(\frac{p'}{q'}\right)$$

for all $\frac{p'}{q'} \neq \frac{p}{q}$, so the set of minimal vectors for the quadratic form Q is exactly

$$\mathrm{Min}(Q) = \left\{\infty, \frac{p}{q}\right\}.$$

□

2.3 Binary Hermitian Forms and Hyperbolic 3-Space

We now move to the imaginary quadratic case.

Definition 2.7. Let F be an imaginary quadratic field. A *binary Hermitian form* over F is a map $f : F^2 \to \mathbb{R}$ given by

$$f(x, y) = ax\overline{x} + bx\overline{y} + \overline{b}\overline{x}y + cy\overline{y},$$

where $a, c \in \mathbb{R}$ and $b \in F$. If f maps F^2 to $\mathbb{R}_{>0}$, we say that f is *positive definite.*

To each binary Hermitian form f we can associate a 2×2 complex Hermitian matrix:

Definition 2.8. For a binary Hermitian form $f(x, y) = ax\overline{x} + bx\overline{y} + \overline{b}\overline{x}y + cy\overline{y}$, the *matrix of* f is the 2×2 Hermitian matrix M_f with entries in $\mathbb{C}$ given by

$$M_f = \begin{bmatrix} a & b \\ \overline{b} & c \end{bmatrix},$$

so that $f(x, y) = \begin{bmatrix} \overline{x} & \overline{y} \end{bmatrix} M_f \begin{bmatrix} x \\ y \end{bmatrix}$.

The value a Hermitian form f takes on the ordered pair (x_0, y_0) is equal to the result of $v^* M_f v$, where $v = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ and $(*)$ denotes conjugate transpose. Note that if f is positive definite, then $f(x, y) > 0$ for all pairs (x, y) other than $(0, 0)$.

Definition 2.9. The *upper half space*, denoted by $\mathbb{H}^3$, is a model of three-dimensional hyperbolic geometry consisting of $\mathbb{C} \times \mathbb{R}_{>0}$,

$$\mathbb{H}^3 = \{(z, t) : z \in \mathbb{C}, t \in \mathbb{R}_{>0}\}.$$

There is a correspondence between points in hyperbolic 3-space and positive definite binary Hermitian forms. The group $\mathrm{GL}_2(\mathbb{C})$ acts transitively on $\mathbb{H}^3$ by

$$\sigma \cdot (z,t) = \left(\frac{(\overline{d} - \overline{cz})(az-b) - t^2\overline{c}a}{|cz-d|^2 + t^2|c|^2}, \frac{|\Delta| t}{|cz-d|^2+t^2|c|^2} \right),$$

where $\sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{C})$ and $\Delta = \det \sigma = ad - bc$ [Swa71]. The stabilizer of the point $(0,1)$ is isomorphic to the special unitary group $\mathrm{SU}(2)$.

We identify points in the upper half space with the cone of positive definite binary Hermitian forms (mod scaling by positive real numbers) by the map

$$(z,t) \mapsto \begin{bmatrix} 1 & -z \\ -\overline{z} & |z|^2 + t^2 \end{bmatrix}.$$

The action on $\mathbb{H}^3$ is compatible with the left action by $\mathrm{GL}_2(\mathbb{C})$ on positive definite binary Hermitian forms; for a Hermitian form f and $v \in \mathbb{C}^2$,

Definition 2.10 ([Swa71])**.** If $\lambda, \mu \in \mathcal{O}_F$ generate the unit ideal and $\mu \neq 0$, let $S_{\lambda,\mu}$ denote the hemisphere in $\mathbb{H}^3$ given by $|\mu z - \lambda|^2 + t^2|\mu|^2 = 1$. This is a Euclidean hemisphere with center $\left(\frac{\lambda}{\mu}, 0\right)$ and radius $\frac{1}{|\mu|}$. Let B be the set of points in $\mathbb{H}^3$ which lie above or on $S_{\lambda,\mu}$; i.e., B is the set of all $(z,t) \in \mathbb{H}^3$ satisfying the inequality $|\mu z - \lambda|^2 + t^2|\mu|^2 \geq 1$ for all $\lambda, \mu \in \mathcal{O}_F$ which generate the unit ideal.

Let F be an imaginary quadratic field with ring of integers $\mathcal{O}_F$. Let Ω be a fundamental domain for the group of translations by elements of $\mathcal{O}_F$, and Ω' be the set of all points $(z,t) \in \mathbb{H}^3$ with $z \in \Omega \cap F$ such that $(z,t) \in B$ as defined above. Note that by our correspondence between Hermitian forms and points in $\mathbb{H}^3$, the points in B represent all positive definite binary Hermitian forms f for which $f(1,0)$ is the minimum of f.

Theorem 2.11 ([Swa71] 3.13)**.** *There are only a finite number of $\lambda, \mu \in \mathcal{O}_F$ with $(\lambda, \mu) = \mathcal{O}_F$ such that $\Omega' \cap S_{\lambda,\mu} \neq \emptyset$.*

We can visualize the layout of the hemispheres at the boundary of $\mathbb{H}^3$ by setting $t = 0$ and drawing the resulting circles in the complex plane. There are five imaginary quadratic fields whose rings of algebraic integers are Euclidean, namely $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$, $\mathbb{Q}(\sqrt{-3})$, $\mathbb{Q}(\sqrt{-7})$, and $\mathbb{Q}(\sqrt{-11})$. For these five fields, every point in F lies within 1 of an integer, thus F is covered by hemispheres over integer points. The hemispheres covering Ω for each of the Euclidean cases are shown in Figures 2.2-2.6.

When the ring of integers $\mathbb{Z}[\omega]$ is a principal ideal domain, even if it is not Euclidean, the fundamental region Ω can be covered by hemispheres over principal cusps [Swa71]. The imaginary quadratic fields for which the ring of integers is a P.I.D. are exactly

Figure 2.2. $\mathbb{Q}(\sqrt{-1})$. The Gaussian integers $\mathbb{Z}[\sqrt{-1}]$ are a Euclidean ring.

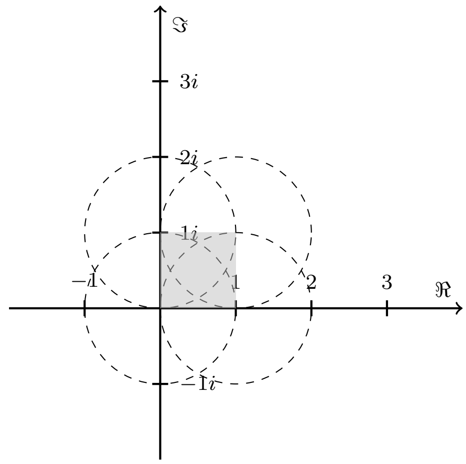


Figure 2.3. $\mathbb{Q}(\sqrt{-2})$. The ring of integers $\mathbb{Z}[\omega]$, where $\omega = \sqrt{-2}$, is Euclidean.

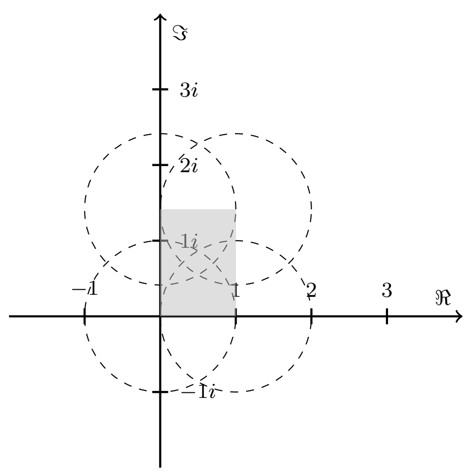


Figure 2.4. $\mathbb{Q}(\sqrt{-3})$. The ring of integers $\mathbb{Z}[\omega]$, where $\omega = \frac{1+\sqrt{-3}}{2}$, is Euclidean.

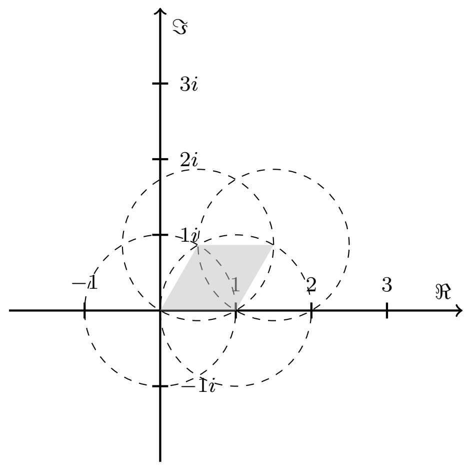


Figure 2.5. $\mathbb{Q}(\sqrt{-7})$. The ring of integers $\mathbb{Z}[\omega]$, where $\omega = \frac{1+\sqrt{-7}}{2}$, is Euclidean.

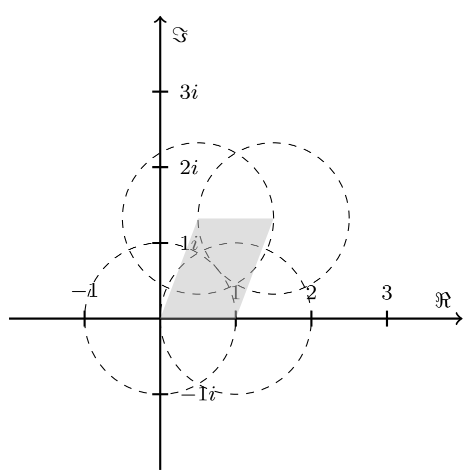
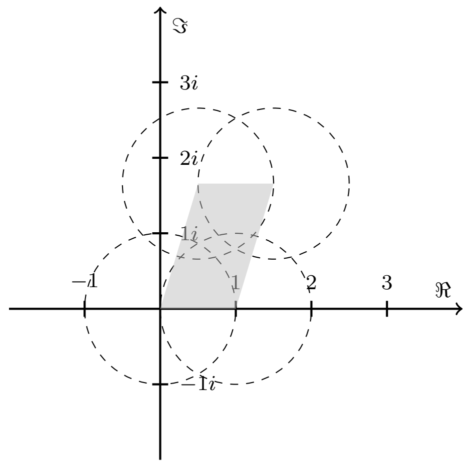
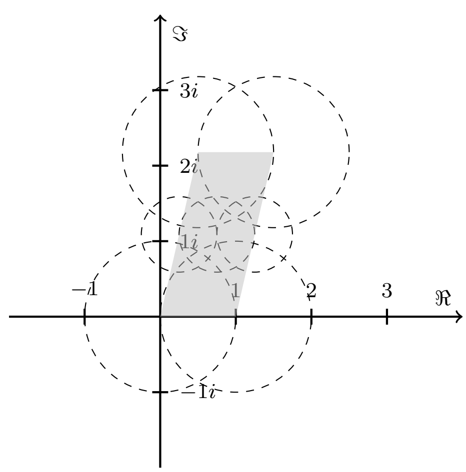


Figure 2.6. $\mathbb{Q}(\sqrt{-11})$. The ring of integers $\mathbb{Z}[\omega]$, where $\omega = \frac{1+\sqrt{-11}}{2}$, is Euclidean.



the fields with class number $h(F) = 1$; there exist four such fields in addition to the five Euclidean cases, namely $\mathbb{Q}(\sqrt{-19})$, $\mathbb{Q}(\sqrt{-43})$, $\mathbb{Q}(\sqrt{-67})$, $\mathbb{Q}(\sqrt{-163})$ [Sta69]. A covering of the fundamental region Ω by principal hemispheres for $\mathbb{Q}(\sqrt{-19})$ is shown in Figure 2.7. Note that the integer hemispheres are no longer enough to cover Ω.

Figure 2.7. $\mathbb{Q}(\sqrt{-19})$. The ring of integers $\mathbb{Z}[\omega]$, where $\omega = \frac{1+\sqrt{-19}}{2}$, is a non-Euclidean principal ideal domain.



2.4 Voronoi Tessellations of $\mathbb{H}^3$

In the classical case, the Farey tessellation of the upper half plane coincides with the Voronoi tessellation arising from the theory of quadratic forms. Here we discuss a generalization of Voronoi that applies to Hermitian forms, following closely the exposition of [STY21].

Let F be a number field with ring of integers $\mathcal{O}_F$. The space of positive definite Hermitian forms over F form an open cone in a real vector space. There is a natural decomposition of this cone into polyhedral cones corresponding to the facets of the Voronoi-Koecher polyhedron Π [Koe60, Ash77].

The Voronoi complex is the result of a polyhedral reduction theory for Γ developed by Ash [AMRT10, Ch. II] and Koecher [Koe60], generalizing Voronoi's work [Vor08] on perfect quadratic forms over $\mathbb{Q}$. The top-dimensional cells in the Voronoi complex are in bijection with Γ-equivalence classes of perfect n-ary forms.

Fix a square-free positive integer d. Let F be the imaginary quadratic field $F = \mathbb{Q}(\sqrt{-d})$, with ring of algebraic integers $\mathcal{O}_F$. Then F has discriminant $\Delta = -4d$ if $d \equiv 1, 2 \bmod 4$, and $\Delta = -d$ otherwise. The ring of integers $\mathcal{O}_F$ is equal to $\mathbb{Z}[\omega]$, where

$$\omega = \begin{cases} \sqrt{-d} & \text{if } d \equiv 1, 2 \bmod 4 \\ \frac{1+\sqrt{-d}}{2} & \text{if } d \equiv 3 \bmod 4. \end{cases}$$

We fix a complex embedding $F \hookrightarrow \mathbb{C}$, identify F with the image, and extend this identification to vectors and matrices. The notation $\bar{\cdot}$ denotes complex conjugation on $\mathbb{C}$, the non-trivial Galois automorphism on F.

Let V be the 4-dimensional real vector space of 2×2 Hermitian matrices with complex coefficients,

$$V = \left\{ \begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix} : a, c \in \mathbb{R}, b \in \mathbb{C} \right\}.$$

Let $C \subset V$ denote the subset of positive definite matrices. Then C is an open cone with a boundary consisting of semi-definite Hermitian forms. The minimal vectors of a Hermitian form can be represented by elements on the boundary of C via the map q defined below.

Using the chosen complex embedding of F, we view V_F, the 2×2 Hermitian matrices with entries in F, as a subset of V. Define a map $q \colon \mathcal{O}_F^2 \setminus \{0\} \to V$ by $q(x) = x\bar{x}^t$. For each $x \in \mathcal{O}_F^2$, we have that $q(x)$ is on the boundary of C. Let C^* denote the union of C and the image of q.

The group $\mathrm{GL}_2(\mathbb{C})$ acts on V by $g \cdot A = gA\bar{g}^t$. The image of C in the quotient of V by positive homotheties can be identified with hyperbolic 3-space $\mathbb{H}^3$. The image of q in this quotient is identified with $\mathbb{P}^1(F) = F \cup \{\infty\}$, the set of cusps.

Each $A \in V$ defines a Hermitian form $A[x] = \bar{x}^t Ax$, for $x \in \mathbb{C}^2$. Using the chosen complex embedding of F, we can view $\mathcal{O}_F^2$ as a subset of $\mathbb{C}^2$.

Definition 2.12. For $A \in C$, we define the *minimum of* A to be the minimum value obtained by evaluation on integer vectors,

$$\min(A) := \min_{v \in \mathcal{O}_F^2 \setminus \{0\}} \{A[v]\}.$$

Note that $\min(A) > 0$ since A is positive definite. A vector $v \in \mathcal{O}_F^2$ is called a *minimal vector of* A if $A[v] = \min(A)$. We let $\mathrm{Min}(A)$ denote the set of minimal vectors of A.

Since $\mathcal{O}_F^2$ is discrete in the topology of $\mathbb{C}^2$, the set $\mathrm{Min}(A)$ is finite. A minimal vector $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in \mathcal{O}_F^2$ generates an ideal $(\alpha, \beta) \subseteq \mathcal{O}_F$ that has minimal norm among ideals in its class in the class group of F.

Definition 2.13. We say a Hermitian form $A \in C$ is a *perfect Hermitian form over* F if

$$\mathrm{span}_{\mathbb{R}} \{q(v) : v \in \mathrm{Min}(A)\} = V.$$

The above definition is equivalent to the statement that a Hermitian form A is *perfect* if $\mathrm{Min}(A)$ determines A up to scaling by $\mathbb{R}^+$.

Definition 2.14. A *polyhedral cone* in V is a subset σ of the form

$$\sigma = \left\{ \sum_{i=1}^{n} \lambda_i q(v_i) : \lambda_i \geq 0 \right\},$$

where $v_1, v_2, \ldots, v_n$ are non-zero vectors in $\mathcal{O}_F^2$.

Definition 2.15. A set of polyhedral cones S forms a *fan* if the following two conditions hold:

1. If σ is in S and τ is a face of σ, then τ is in S.

2. If σ and σ' are in S, then $\sigma \cap \sigma'$ is a common face of σ and σ'.

Note that a face here can be of codimension higher than 1.

Theorem 2.16. *There is a fan S in V with $\mathrm{GL}_2(\mathcal{O}_F)$-action such that the following hold.*

1. *There are only finitely many $\mathrm{GL}_2(\mathcal{O}_F)$-orbits in S.*

2. *Every $y \in C$ is contained in the interior of a unique cone in S.*

3. *Any cone $\sigma \in S$ with non-trivial intersection with C has finite stabilizer in $\mathrm{GL}_2(\mathcal{O}_F)$.*

The 4-dimensional cones in S are in bijection with perfect forms over F.

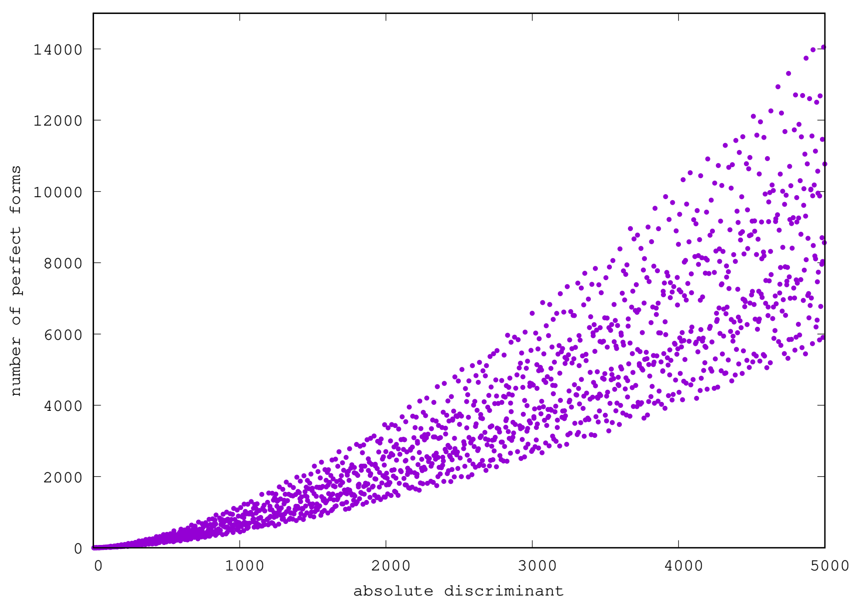
The bijection is explicit and allows one to compute the structure of S using a modification of Voronoi's algorithm [DSGG+16, §2, §6]. Specifically, σ is a 4-dimensional cone in S if and only if there exists a perfect Hermitian form A such that

$$\sigma = \left\{ \sum_{v \in \mathrm{Min}(A)} \lambda_v q(v) : \lambda_v \geq 0 \right\}.$$

Modulo positive homotheties, the fan S descends to a $\mathrm{GL}_2(\mathcal{O}_F)$-tessellation of $\mathbb{H}^3$ by ideal polytopes.

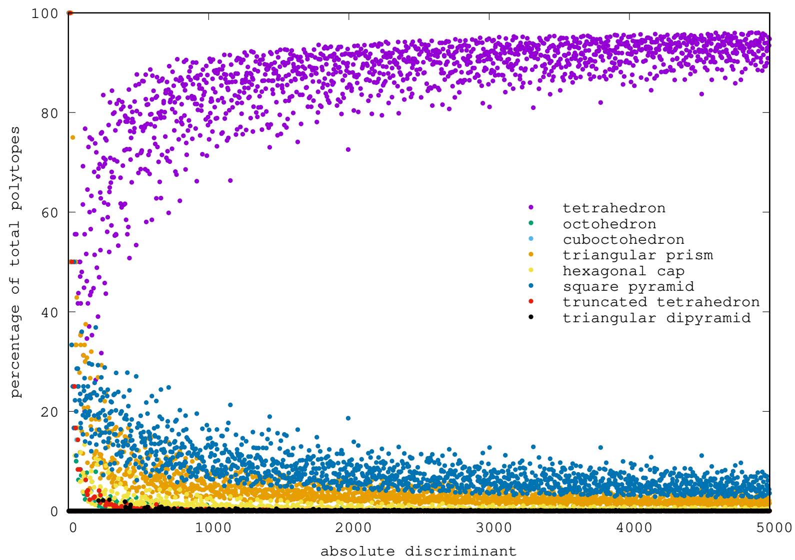
In joint work with Thalagoda and Yasaki ([STY21]) we classified the perfect forms for all imaginary quadratic fields F of absolute discriminant up to 5000, extending previous explicit computations [Cre84] and [Yas10]. See Figure 2.8 for a plot of $N_{perf}(F)$ as a function of the discriminant of F.

Figure 2.8. Number of Perfect Forms for Imaginary Quadratic Fields. The number of perfect forms $N_{perf}(F)$, indexed by absolute discriminant of F up to 5000.



The following theorem gives a sharp bound ensuring that the configurations of minimal vectors of perfect binary Hermitian forms over imaginary quadratic fields do not get arbitrarily complicated. In particular, the number of minimal vectors is bounded, independent of the field, so there are only a finite number of combinatorial types of ideal polytopes arising in the Voronoi tessellation of $\mathbb{H}^3$.

Figure 2.9. Observed Polytope Types as a Percentage of Total. Observed polytope types as a percentage of the total number of polytopes, indexed by absolute discriminant of F.



Theorem 2.17. *Let A be a positive definite binary Hermitian form over an imaginary quadratic field. Then*

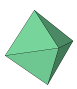
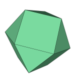
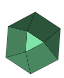
$$\#\mathrm{Min}(A) \leq 12.$$

Proof. See [STY21]. $\square$

Since the minimal vectors of each perfect form map to vertices of their corresponding polytopes in the cone, and there are finitely many combinatorial types of polytopes with 12 or fewer vertices, only finitely many types of polytopes can arise in a tessellation of $\mathbb{H}^3$ as we vary the discriminant of the imaginary quadratic field. Data from [Dil96] indicates that there are more than six million combinatorial types of 3-dimensional polytopes. However, in our range of computations, only 8 distinct combinatorial types of polytopes were observed, as shown in Table 2.1.

As the discriminant increases, the total number of polytopes increases and appears to be dominated by tetrahedra. The types of observed polytopes as a percentage of the total number of polytopes computed (up to absolute discriminant 5000) are plotted in Figure 2.9.

Table 2.1. Combinatorial Types of Polytopes Observed. The total number of fields witnessing each type of polytope is listed, as well as the total percentage of each polytope type observed in the current range of computation.

	Polytope Type	Number of Fields	Percentage of Polytopes
	Tetrahedron	1504	91.524
	Octahedron	912	0.066
	Cuboctahedron	16	0.005
	Triangular prism	1511	2.416
	Hexagonal cap	1358	0.199
	Square pyramid	1506	5.764
	Truncated tetrahedron	60	0.007
	Triangular dipyramid	416	0.019

2.5 Complications in Higher Class Numbers

Generalized Voronoi tessellations for imaginary quadratic fields connect nicely with hemisphere theory when the ring of integers $\mathcal{O}_F$ is a principal ideal domain, i.e., when the class number $h(F)$ is 1.

Definition 2.18. A *fractional ideal* I of $\mathcal{O}_F$ is a nonzero submodule of F such that there exists a nonzero integer $d \in \mathbb{Z}$ with dI an ideal of $\mathcal{O}_F$. If a fractional ideal I is an ideal of $\mathcal{O}_F$, then I is an *integral ideal.* An ideal (fractional or not) is said to be a *principal ideal* if there exists $x \in F$ such that $I = x\mathcal{O}_F$. Finally, $\mathcal{O}_F$ is a *principal ideal domain (PID)* if every ideal of $\mathcal{O}_F$ is a principal ideal.

Definition 2.19. Let I be a fractional ideal of $\mathcal{O}_F$. We say that I is *invertible* if there exists a fractional ideal J of $\mathcal{O}_F$ such that $IJ = \mathcal{O}_F$.

Note that every fractional ideal in the ring of integers of a number field is invertible. The set of fractional ideals of $\mathcal{O}_F$ form a group, and the set of principal ideals form a subgroup.

Definition 2.20. Let F be a number field. The *ideal class group* $\mathcal{C}\ell_F$ or *class group* of F is the quotient of the group of fractional ideals J_F by the subgroup of principal ideals P_F.

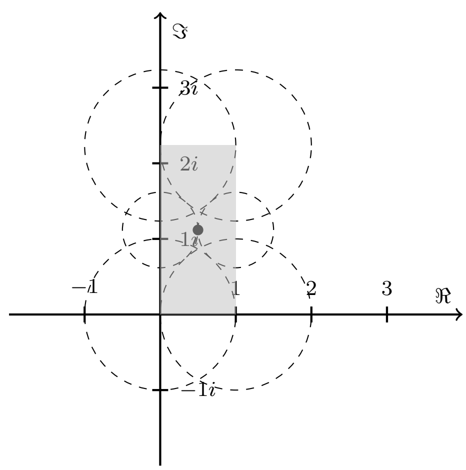
Theorem 2.21 ([Neu92])**.** *The class group $\mathcal{C}\ell_F$ is finite. Its order*

$$h(F) = [J_F : P_F]$$

is called the class number *of F.*

When the class number $h(F)$ is greater than 1, F has singular points; that is, the cusps are no longer all equivalent up to $\mathrm{GL}_2(\mathcal{O}_F)$. Equivalence classes of cusps up to the action are parameterized by elements in the class group, so each element beyond the trivial one corresponds to an equivalence class of singular points. The singular points in F give rise to hemispheres but cannot be used for inversions, as there is no matrix in $\mathrm{GL}_2(\mathcal{O}_F)$ that maps a singular point to ∞. The connection between hemispheres and Voronoi edges – the argument that a point in $\mathbb{H}^3$ lying on a given principal hemisphere but outside of all other principal hemispheres gives rise to an edge in the Voronoi tessellation – no longer holds, since the point in question may lie on or under a hemisphere corresponding to a singular point. For example, if $F = \mathbb{Q}(\sqrt{-5})$, with class number $h(F) = 2$, using equivalence classes of edges in the Voronoi tessellation to generate hemispheres yields the partial covering of Ω shown in Figure 2.10. There are points in F near the singular point $\frac{1+w}{2}$ which are not covered by any of these principal hemispheres. This example is discussed in greater detail in Section 3.2.3.

Figure 2.10. $\mathbb{Q}(\sqrt{-5})$ with a Singular Point. The ring of integers $\mathbb{Z}[\omega]$, where $\omega = \sqrt{-5}$, is not a principal ideal domain. The point $\frac{1+w}{2}$ is a singular point.



Chapter 3

Generalized Continued Fractions

3.1 The Generalized Continued Fraction Algorithm

Definition 3.1. Let F be the field of fractions of an integral domain R, and let $z \in F$. An *inversion relative to* z is a map $u : F \to \mathbb{P}^1(F)$ for which $u(z) = \infty$.

When $R = \mathbb{Z}$ and $F = \mathbb{Q}$, for example, we may take inversion relative to 0 to be the map $z \mapsto -\frac{1}{z}$. We compute the image under inversion by means of matrix-vector multiplication, where an element $\frac{\alpha}{\beta} \in F$ is represented by the column vector $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in \mathcal{O}_F^2$, the point at infinity is a column vector with second entry 0, and the matrix representation of an inversion is an element in $\mathrm{GL}_2(\mathcal{O}_F)$. In general, an inversion relative to a particular cusp is not unique; for example, both $u_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ are inversions relative to 0.

Since $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \in \mathrm{GL}_2(\mathcal{O}_F)$ for all $x \in \mathcal{O}_F$, we can find a fundamental domain for the action of $\mathrm{GL}_2(\mathcal{O}_F)$ that is contained within the region Ω with vertices $\{0, 1, \omega, \omega + 1\}$.

Let $\{v_1, v_2, \ldots, v_n\}$ be the finite list of rational points determining the hemispheres needed to cover Ω, and let $U = \{u_1, u_2, \ldots, u_n\}$ be a finite set of inversions such that u_j is an inversion relative to v_j for each $j = 1, 2, \ldots, n$.

Let $\alpha \in F$, and let S be the finite set of singular points of F contained in Ω.

Following [Ara10] we define a pseudo-Euclidean function on the set of cusps $\mathbb{P}^1(F)$ as follows:

$$\psi\left(\frac{\lambda}{\mu}\right) = \begin{cases} |\mu|^2 & \text{if } \frac{\lambda}{\mu} \text{ is principal} \\ 0 & \text{if } \frac{\lambda}{\mu} = \infty \\ \frac{N\langle\mu\rangle}{N\langle\lambda,\mu\rangle} & \text{if } \frac{\lambda}{\mu} \in S \text{ is singular.} \end{cases}$$

Note that $\psi(\alpha) = 1$ for all $\alpha \in \mathcal{O}_F$.

In the cases where the set of singular points S is empty, the algorithm described in Theorem 1.15 may be implemented in Magma and used to compute generalized continued fraction convergents:

Algorithm 3.2 (Generalized Continued Fractions)**.** *The list of convergents from ∞ to α is computed from the matrix form of the edges between them.*

Input: $\alpha \in F$, where $h(F) = 1$

Output: list of convergents following edges of the tessellation

1. Set up inversions: compute list of principal cusps corresponding to inversion hemispheres necessary to cover Ω; sort by increasing norm of denominator; compute matrix sending each cusp to infinity.
2. While α is not infinity, translate so $\alpha_0 \in \Omega$; record the translation matrix and its inverse in separate lists.
3. For principal cusps in the inversion list, check whether α_0 lies under the inversion hemisphere; break when true, record the inversion matrix and its inverse.
4. Invert α_0 with selected matrix, update α; if α is not infinity, return to step 2.
5. Compute products of translations and inversions: for each round of translation and inversion, compute $T_i^{-1}U_i^{-1}$.
6. Compute continued fraction convergents: begin with identity matrix, append products one at a time from the previous step on the right, multiply by the vector representing infinity. The first convergent is $T_1^{-1}U_1^{-1}\infty$, the second is $T_1^{-1}U_1^{-1}T_2^{-1}U_2^{-1}\infty$, and so on. The last convergent is α itself.

The computation of the list of convergents to α is done in an analogous way to the computation of continued fraction convergents in the classical case, using the output of the (psuedo-)Euclidean algorithm.

Recall for the classical case, the pair $\left\{\frac{p}{q}, \frac{p'}{q'}\right\}$ determines an edge in the Farey tessellation of $\mathcal{H}$ if and only if $\det\left(\begin{bmatrix} p & p' \\ q & q' \end{bmatrix}\right) = \pm 1$. In the continued fraction expansion of $\alpha \in \mathbb{Q}$, for each adjacent pair of convergents $\frac{p}{q}, \frac{p'}{q'}$ we have $pq' - p'q = \pm 1$, thus each adjacent pair of convergents represents an edge in the tessellation.

Theorem 3.3. *Let F be an imaginary quadratic field whose ring of integers is Euclidean. Then each adjacent pair of convergents in the finite list returned by the generalized continued fraction algorithm determines an edge in the Voronoi tessellation of $\mathbb{H}^3$ for F.*

Proof. Let F be an imaginary quadratic field whose ring of integers is Euclidean. Then the class number of F is 1, and every element of F is equivalent to ∞. Let $\alpha \in F$. Then there exists $\gamma \in \mathrm{GL}_2(\mathcal{O}_F)$ such that $\gamma \cdot \alpha = \infty$, so the pseudo-Euclidean algorithm in Theorem 1.15 generates a list of elements of $\mathrm{GL}_2(\mathcal{O}_F)$ such that

$$\infty = U_m T_m U_{m-1} T_{m-1} \cdots U_2 T_2 U_1 T_1 \alpha,$$

where the U_j are inversions (not necessarily unique) in the set U defined above, and the T_j are translations by elements in $\mathcal{O}_F$. Reversing the process, we have

$$\alpha = T_1^{-1} U_1^{-1} T_2^{-1} U_2^{-1} \cdots T_{m-1}^{-1} U_{m-1}^{-1} T_m^{-1} U_m^{-1} \infty.$$

For $j = 1, 2 \dots, m-1$, we obtain the j^{th} convergent to α by deleting the sequence $T_{j+1}^{-1} \cdots U_m^{-1}$ from the above product; the m^{th} convergent is α itself. For example, the first convergent is $T_1^{-1} U_1^{-1} \infty$, and the fourth is

$$T_1^{-1} U_1^{-1} T_2^{-1} U_2^{-1} T_3^{-1} U_3^{-1} T_4^{-1} U_4^{-1} \infty.$$

Note that each pair of adjacent convergents in the list is of the form $\{g \cdot \infty, g \cdot \beta\}$, with $g \in \mathrm{GL}_2(\mathcal{O}_F)$ and $\beta = T_k^{-1} U_k^{-1} \infty$ for some k. Then β is equivalent up to translation by R to one of the points v from which the finite set of inversions was obtained. Since this is a Euclidean case, these points are in R, thus $\beta \in R$.

Since $\{\infty, 0\}$ is an edge in the Voronoi tessellation, and $\beta \in R$ implies β is equivalent to 0 up to translation, we have that $\{\infty, \beta\}$ is an edge in the Voronoi tessellation. Then since the action of $\mathrm{GL}_2(\mathcal{O}_F)$ takes edges to edges, $\{g \cdot \infty, g \cdot \beta\}$ is an edge in the tessellation. $\square$

Lemma 3.4. *The finite set of inversions for the generalized continued fraction algorithm can be generated using only principal cusps.*

In the case where the input $\alpha \in F$ is equivalent to infinity, the orbit of α is covered by principal hemispheres ([Swa71, Ara10], and the algorithm terminates upon reaching infinity. In the case where α is equivalent to a singular point, the orbit of α includes only non-principal cusps, and the algorithm terminates upon reaching a singular point (which is not covered by a principal hemisphere).

Lemma 3.5. *Let F be an imaginary quadratic field of class number 1, and let $\frac{\lambda}{\mu} \in \mathbb{P}^1(F)$ be a principal cusp. If a point $(z,t) \in \mathbb{H}^3$ lies on the hemisphere determined by $\frac{\lambda}{\mu}$ and does not lie under any other principal hemisphere, then there exists a Hermitian form whose set of minimal vectors is* $\left\{ \begin{bmatrix} \lambda \\ \mu \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$.

In the class number 1 cases, since F has no singular points, this result follows from the previous result for principal hemispheres.

Theorem 3.6. *Let F be an imaginary quadratic field with class number 1. Then each adjacent pair of convergents in the finite list returned by the generalized continued fraction algorithm determines an edge in the Voronoi tessellation of $\mathbb{H}^3$ for F.*

Proof. Let F be an imaginary quadratic field with $h(F) = 1$ and with ring of integers R. Let $\alpha \in F$. Since the class number is 1, α is equivalent to ∞. Then there exists $\gamma \in \mathrm{GL}_2(\mathcal{O}_F)$ such that $\gamma \cdot \alpha = \infty$, so the pseudo-Euclidean algorithm in Theorem 1.15 generates a list of elements of $\mathrm{GL}_2(\mathcal{O}_F)$ such that

$$\infty = U_m T_m U_{m-1} T_{m-1} \cdots U_2 T_2 U_1 T_1 \alpha,$$

where the U_j are inversions (not necessarily unique) in the set U defined above, and the T_j are translations by elements in $\mathcal{O}_F$. Reversing the process, we have

$$\alpha = T_1^{-1} U_1^{-1} T_2^{-1} U_2^{-1} \cdots T_{m-1}^{-1} U_{m-1}^{-1} T_m^{-1} U_m^{-1} \infty.$$

For $j = 1, 2 \ldots, m-1$, we obtain the j^{th} convergent to α by deleting the sequence $T_{j+1}^{-1} \cdots U_m^{-1}$ from the above product; the m^{th} convergent is α itself. For example, the first convergent is $T_1^{-1} U_1^{-1} \infty$, and the fourth is

$$T_1^{-1} U_1^{-1} T_2^{-1} U_2^{-1} T_3^{-1} U_3^{-1} T_4^{-1} U_4^{-1} \infty.$$

Note that each pair of adjacent convergents in the list is of the form $\{g \cdot \infty, g \cdot \beta\}$, with $g \in \mathrm{GL}_2(\mathcal{O}_F)$ and $\beta = T_k^{-1} U_k^{-1} \infty$ for some k. Then β is equivalent up to translation by R to one of the points v from which the finite set of inversions was obtained. These inversions were computed form the Voronoi data and correspond to edges in the tessellation, thus $\{\infty, \beta\}$ is an edge in the Voronoi tessellation. Then since the action of $\mathrm{GL}_2(\mathcal{O}_F)$ takes edges to edges, $\{g \cdot \infty, g \cdot \beta\}$ is an edge in the tessellation.

□

3.2 Examples

3.2.1 Class Number 1, Euclidean

Let $F = \mathbb{Q}(\sqrt{-7})$, and let $\alpha \in F$ be the element $\frac{-77\omega-97}{-13}$. Using Algorithm 3.2, the continued fraction convergents to α are given by

$$\frac{1}{0}, \frac{6\omega+7}{1}, \frac{-18\omega-22}{-3}, \frac{52\omega-21}{3\omega+2}, \frac{3\omega+61}{-2\omega+5}, \frac{-9\omega+46}{-2\omega+3}, \frac{-77\omega-97}{-13}.$$

3.2.2 Class Number 1, Non-Euclidean

Let $F = \mathbb{Q}(\sqrt{-43})$, and let $\alpha \in F$ be the element $\frac{71\omega+33}{63}$. Using Algorithm 3.2, the continued fraction convergents to α are given by

$$\frac{1}{0}, \frac{\omega+1}{1}, \frac{4\omega-11}{\omega+2}, \frac{26}{-2\omega+3}, \frac{-4\omega-15}{\omega-5}, \frac{6\omega-63}{5\omega-2}, \frac{71\omega+33}{63}.$$

3.2.3 Class Number 2

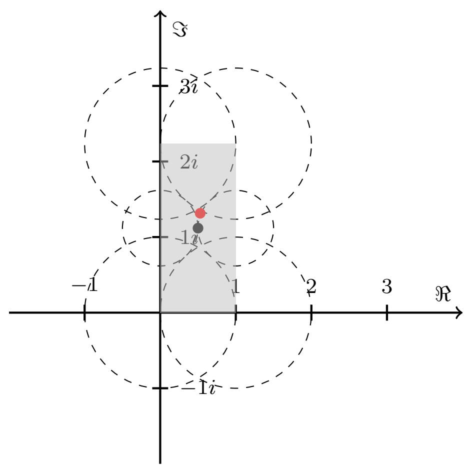
When the class number of F is greater than 1, there exist points in F (the so-called singular points) that are not equivalent to infinity modulo the action of $\mathrm{GL}_2(\mathcal{O}_F)$. In the upper half space picture, these points still give rise to hemispheres as equality sets for the evaluation of Hermitian forms, but since the points themselves cannot be sent to infinity, these hemispheres do not contribute to the set of inversions used in the generalized continued fraction algorithm. They may, however, contribute minimal vectors to Hermitian forms on the hemispheres over principal cusps, including those necessary to form a covering of the fundamental domain. Since edges in the Voronoi tessellation indicate the existence of Hermitian forms whose minimal vectors arise from exactly two distinct points in $\mathbb{P}^1(F)$, the edges in the tessellation for a field of class number 2 or higher may not provide a sufficient number of hemispheres to cover the fundamental domain.

For example, let $F = \mathbb{Q}(\sqrt{-5})$. From the tables of examples computed by [Ara10], a fundamental region for the action of $\mathrm{GL}_2(\mathcal{O}_F)$ can be covered by hemispheres over principal cusps $\frac{\lambda}{\mu}$, where $|\mu|^2 \leq 20$. The singular points in this case (up to the action) are given by $\left\{\pm\frac{1}{2} + \frac{\sqrt{-5}}{2}\right\}$.

Let Ω be the 1-by-ω box with lower left vertex 0, a fundamental region for the group of translation by elements of $\mathcal{O}_F$.

Using Magma to compute the singular points and inversion hemispheres in Ω for $\mathbb{Q}(\sqrt{-5})$ from our Voronoi data, we find the set of singular points S_F up to translation

Figure 3.1. $\mathbb{Q}(\sqrt{-5})$ with an Uncovered Principal Cusp. The point $\frac{9+10\omega}{17}$ is a principal cusp, but is not covered by the principal hemispheres corresponding to Voronoi edges.



contains only one cusp,

$$S_F = \left\{ \begin{bmatrix} 1+\omega \\ 2 \end{bmatrix} \right\},$$

which is consistent with the cited result above, and the set of principal cusps α for which $\{g\alpha, \infty\}$ is an edge in the Voronoi tessellation for some $g \in \mathrm{GL}_2(\mathcal{O}_F)$ is

$$\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \omega \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1+\omega \\ 1 \end{bmatrix}, \begin{bmatrix} \omega \\ 2 \end{bmatrix}, \begin{bmatrix} 2+\omega \\ 2 \end{bmatrix} \right\} \subset \Omega.$$

Note that this includes principal cusps with denominators of norm no more than 4, which does not meet the bound established by Aranes. These hemispheres nearly cover Ω, but there exist points close to the singular point in the center which lie under the hemisphere generated by that singular point. Since every point in $F \setminus S_F$ is covered by a principal hemisphere ([Swa71]), there exist points in $\mathbb{H}^3$ above these small regions whose associated Hermitian forms have more than two minimal vectors. Then the set of principal hemispheres generated by our Voronoi edge data is not enough to cover Ω, and our generalized continued fraction algorithm will break if it reaches a point in the uncovered region.

Figure 3.1 shows that not every principal cusp in this class number 2 case is covered by principal hemispheres arising from the Voronoi tessellation. The point $\alpha = \frac{9+10\omega}{17}$ is principal, yet does not lie under any of the six computed hemispheres; α does, however, lie under the hemisphere over the singular point $\frac{1+\omega}{2}$.

Chapter 4

Future Directions

Modular forms are holomorphic functions on the complex upper half plane $\mathcal{H}$ which, under the action of the *modular group* $\mathrm{SL}_2(\mathbb{Z})$, transform in a predictable, nearly invariant way [Con16]. Finite-index subgroups of $\mathrm{SL}_2(\mathbb{Z})$, called *congruence subgroups*, also give rise to modular forms for their respective actions.

Definition 4.1 ([Con16]). Let k be an integer. A *modular form of weight* k for $\mathrm{SL}_2(\mathbb{Z})$ is a function $f : \mathcal{H} \to \mathbb{C}$ such that

(i) f is holomorphic on $\mathcal{H}$;

(ii) $f\left(\dfrac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and all $z \in \mathcal{H}$;

(iii) f is "holomorphic at infinity," that is, the values $f(z)$ are bounded as $\mathrm{Im}(z) \to \infty$.

Property *(ii)* is often called the *modularity condition*. Although $\mathrm{SL}_2(\mathbb{Z})$ is an infinite group, it is finitely generated, so we may verify the modularity condition for a particular modular form of a given weight simply by verifying it on the generators of $\mathrm{SL}_2(\mathbb{Z})$.

Proposition 4.2. *The matrix group* $\mathrm{SL}_2(\mathbb{Z})$ *is generated by the matrices* S *and* T*,*

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \textit{and} \quad T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

See [DS05] or [Con16] for detailed proofs.

To further investigate the significance of the modularity condition, we compute a few examples.

Example 4.3. Let f be a modular form of weight k and consider the matrix

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

For any $z \in \mathcal{H}$, the modularity condition on f gives

$$f\left(\frac{1z+1}{0z+1}\right) = (0z+1)^k f(z),$$

thus $f(z+1) = f(z)$. Then every modular form (of any weight) is invariant under horizontal translation by integers.

Example 4.4. Let f be a modular form of weight k and consider the matrix

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

For any $z \in \mathcal{H}$, the modularity condition on f gives

$$f\left(\frac{0z-1}{1z+0}\right) = (1z+0)^k f(z),$$

thus $f\left({-1}/{z}\right) = z^k f(z)$. The appearance of k as an exponent on z shows that the weight k has a significant impact on the symmetry of inversion.

Example 4.5. Let f be a modular form of *odd* weight k and consider the matrix

$$-I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

For any $z \in \mathcal{H}$, the modularity condition on f gives

$$f\left(\frac{-1z+0}{0z-1}\right) = (0z-1)^k f(z),$$

thus $f(z) = (-1)^k f(z)$, and since k is odd, we have $f(z) = -f(z)$ for all z. Then f must be identically zero. It follows that for any odd integer k, the only modular form of weight k is the zero function.

The final condition in the definition of modular form is the notion of holomorphicity "at infinity," where the point at infinity is visualized as being infinitely high up from any position in the upper half plane. To make sense of this, we utilize a change of variables and consider the Fourier expansion (or q-*expansion*) of a modular form.

By the modularity condition, modular forms are translation-invariant: $f(z+1) = f(z)$ for all $z \in \mathcal{H}$, thus the function has a Fourier expansion in terms of $e^{2\pi i z}$. To visualize the transformation of the domain from the half plane picture to the disk picture, consider $z \mapsto e^{2\pi iz}$, for z in a vertical strip of the upper half plane above the interval $[0, 1)$. Then $x + iy \mapsto e^{2\pi i(x+iy)} = e^{2\pi ix}/e^{2\pi y}$, so rotation in the image is determined by x, and scaling is inversely determined by y. For points on the real line, $y = 0$, so the interval $[0, 1)$ maps to the unit circle; since the upper half plane is an open subset of $\mathbb{C}$ which excludes the real line, this effectively results in an open circle in the image. Each vertical line in the half plane model corresponds to a ray through the origin, scaled to a supremum distance from zero of 1 (for $y \to 0$) and infimum distance of 0 (for $y \to \infty$). Thus the point at infinity maps to zero, and the vertical strip over the unit interval maps to the unit disk. Due to translation invariance, every vertical strip maps to the unit disk in the same way. So we may understand the behavior of $f(z)$ on the upper half plane by observing the behavior of the transformed function on $e^{2\pi iz}$ in the unit disk.

Setting $q = e^{2\pi iz}$, we obtain a Laurent series in q centered at zero for the modular form:

$$f(q) = \sum_{n\in\mathbb{Z}} a_n q^n.$$

If the Laurent series is a power series in q (i.e., it has no negative exponents), as is the case for every modular form, then the transformed function is holomorphic at 0, which implies the original function is holomorphic "at infinity."

For certain normalized modular forms, the coefficients in this q-expansion form sequences with various interesting applications to number theory. Computing the expansions is generally difficult, but in the classical case over $\mathbb{Q}$, modular forms can be computed explicitly by means of modular symbols and continued fractions; see Chapter 2 of [Cre97] for details. For Bianchi modular forms, the imaginary quadratic analogues of classical modular forms, one might hope to use generalized continued fractions for these computations. A deeper understanding of the relationship between continued fractions and Voronoi tessellations for imaginary quadratic fields could help push the work beyond the Euclidean cases.

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