AN EXPLORATION OF NATHANSON'S g-ADIC REPRESENTATIONS OF INTEGERS

GREG BELL, AUSTIN LAWSON, NEIL PRITCHARD, AND DAN YASAKI

Abstract. We use Nathanson's g-adic representation of integers to relate metric properties of Cayley graphs of the integers with respect to various infinite generating sets S to problems in additive number theory. If S consists of all powers of a fixed integer g, we find explicit formulas for the smallest positive integer of a given length. This is related to finding the smallest positive integer expressible as a fixed number of sums and differences of powers of g. We also consider S to be the set of all powers of all primes and bound the diameter of Cayley graph by relating it to Goldbach's conjecture.

1. Introduction

Fix an integer $g \geq 2$. Nathanson [9] introduces a g-adic representation of the integers $\mathbb{Z}$ in his investigation of number theoretic analogues of nets in metric geometry. The g-adic representation of an integer provides a method of computing its length in a metric depending on g. Nathanson [8] considers more general sets of integers to investigate questions about the finiteness of the diameter of $\mathbb{Z}$. In this note, we continue these investigation in two different directions. First, we find a formula for the smallest positive integer with a given length (Theorems 3.6 and 3.7). Next, in Section 4 we consider a specific set of integers constructed from primes. We prove that $\mathbb{Z}$ has diameter 3 or 4 in the corresponding metric (Theorem 4.3), and it is 3 assuming Goldbach's Conjecture.

2. Cayley graphs of the integers

Let G be a group. Fix a generating set S for G. Then we can construct a graph $\Gamma = \Gamma(G, S)$, known as the *Cayley graph corresponding to G and S*, by taking the vertices of Γ to be the elements of G and connecting any two vertices g and h by an edge whenever $gs = h$ for some element s in S. We view the graph Γ as a metric space by setting the length of each edge to be 1 and taking the shortest-path metric on Γ; this procedure turns the algebraic object G into a geometric object Γ. In this paper, we investigate different choices of infinite generating sets S when $G = \mathbb{Z}$.

We are primarily interested in generating sets that are closed under additive inverses and are closed under taking powers. The simplest such generating set is the collection

$$S_g = \{\pm 1, \pm g, \pm g^2, \pm g^3, \ldots\}.$$

We denote the Cayley graph $\Gamma(\mathbb{Z}, S_g)$ by C_g. Edges in the graph C_g connect each vertex to infinitely many other vertices, see Figure 1. More generally, let P be a subset of positive

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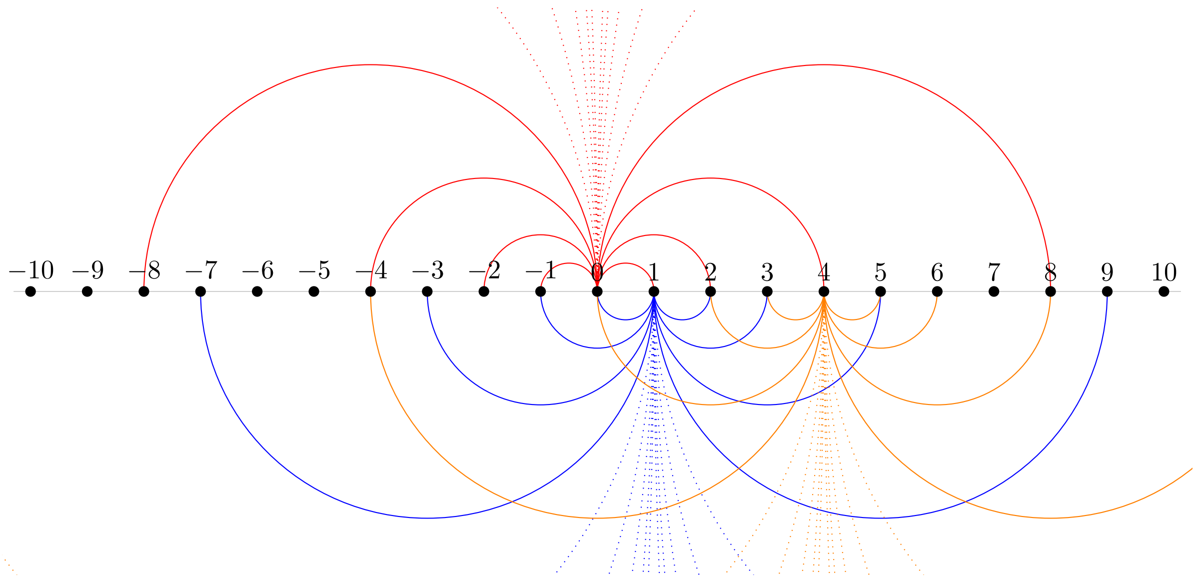

FIGURE 1. Some edges emanating from 0 in red, from 1 in blue, and from 4 in orange in the graph C_2. Each vertex is incident with infinitely many edges. Observe that the distance from 0 to 3 is 2.

integers, and consider generating sets of the form

$$S_P = \bigcup_{g \in P} S_g.$$

Let $C_P = \Gamma(\mathbb{Z}, S_P)$ denote the corresponding Cayley graph.

The study of these graphs leads to an interesting interplay between the geometry of the graph and problems in additive number theory. For example, in C_2 we can ask for the value of the smallest $n > 0$ (in the usual ordering of the integers) at distance d from 0. This is related to the problem of finding the smallest integer that can be expressed as sums and differences of exactly d powers of 2. Looking at Figure 1, one can see that 3 is the smallest positive number that is at distance 2 from 0, since $3 = 2^0 + 2^1$; extrapolating this figure further, one can verify that 11 is the smallest positive integer at distance 3 from 0, since $11 = 2^0 + 2^1 + 2^3$. We investigate this problem in general for $g > 1$ in Section 3.

In Section 4, we investigate C_P for more general subsets P of positive integers. For such graphs, questions about the diameter are already interesting and difficult [8, 1]. We use a covering congruences result of Cohen and Selfridge [4, Theorem 2] together with Helfgott's proof [7, Main Theorem] of the ternary Goldbach's conjecture to show that when P is the set of all primes, the diameter of C_P is either 3 or 4; moreover if Goldbach's conjecture holds, it is 3. We also conduct numerical investigations to narrow the search for the smallest positive length-3 integer in this graph, refining results of Cohen and Selfridge [4] and Sun [11].

3. METRIC PROPERTIES OF C_g

Let $g > 0$ be an integer, and let $C_g = \Gamma(\mathbb{Z}, S_g)$ be the Cayley graph of $\mathbb{Z}$ with the generating set $S_g = \{\pm g^i \mid i \in \mathbb{Z}_{\geq 0}\}$. Let $d_g = d_{S_g}$ denote the corresponding edge-length metric. We denote the distance $d_g(0, n)$ by $\ell_g(n)$ and refer to this as the *g-length* of n.

The following theorems of Nathanson [9] give a method of computing g-length in C_g.

Theorem 3.1 ([9, Theorem 6]). *Let g be an odd integer, $g \geq 3$. Every integer n has a unique representation in the form*

$$n = \sum_{i=0}^{\infty} \epsilon_i g^i$$

such that

(1) $\epsilon_i \in \{0, \pm 1, \pm 2, \ldots, \pm(g-1)/2\}$ *for all nonnegative integers i,*
(2) $\epsilon_i \neq 0$ *for only finitely many nonnegative integers i.*

Moreover, n has g-length

$$\ell_g(n) = \sum_{i=0}^{\infty} |\epsilon_i|.$$

Theorem 3.2 ([9, Theorem 3]). *Let g be an even positive integer. Every integer n has a unique representation in the form*

$$n = \sum_{i=0}^{\infty} \epsilon_i g^i$$

such that

(1) $\epsilon_i \in \{0, \pm 1, \pm 2, \ldots, \pm\frac{g}{2}\}$ *for all nonnegative integers i,*
(2) $\epsilon_i \neq 0$ *for only finitely many nonnegative integers i,*
(3) *if $|\epsilon_i| = \frac{g}{2}$, then $|\epsilon_{i+1}| < \frac{g}{2}$ and $\epsilon_i \epsilon_{i+1} \geq 0$.*

Moreover, n has g-length

$$\ell_g(n) = \sum_{i=0}^{\infty} |\epsilon_i|.$$

For any integer n, Theorems 3.1 and 3.2 give a unique g-adic expression for n that realizes a geodesic path from 0 to n in C_g. Thus there is an $N > 0$ such that $n = \sum_{i=0}^{N} \epsilon_i g^i$, $\epsilon_N \neq 0$, and $\ell_g(n) = \sum_{i=0}^{\infty} |\epsilon_i|$. We call $n = \sum_{i=0}^{N} \epsilon_i g^i$ the *minimal g-adic expansion*, and denote it by

$$[n]_g = [\epsilon_0, \epsilon_1, \ldots, \epsilon_N].$$

Remark 3.3. It is not the case that there is a unique geodesic path from 0 to n. For example, $11 = 1 + 2^1 + 2^3 = -1 - 2^2 + 2^4$.

It is interesting to look at how $\ell_g(n)$ varies as a function of g. See Figure 2. We chose a random number $n = 20{,}233{,}509$, and produced a plot of $y = \ell_g(n)$ for a range of values for g. For g sufficiently large, we have $\ell_g(n) = n$, but it appears that interesting things happen along the way.

Example 3.4. The minimal 5-adic expansion of 46 is $[46]_5 = [1, -1, 2]$, so $46 = 1 - 5 + 2 \cdot 5^2$, and $\ell_5(46) = 1 + 1 + 2 = 4$.

We denote by $\lambda_g(h)$ the smallest positive integer of g-length h in C_g. We find an explicit formula for λ_g in Theorems 3.6 and 3.7 below using Nathanson's g-adic representation [9] of positive integers. The first few values are tabulated in Table 1. We remark that the values of λ_2 show up in The On-Line Encyclopedia of Integer Sequences (OEIS) as A007583, and the values of λ_3 show up as A007051. The sequences of values for λ_p for other primes p did not appear, so the second author has added them. As an example, we chose the prime 19;

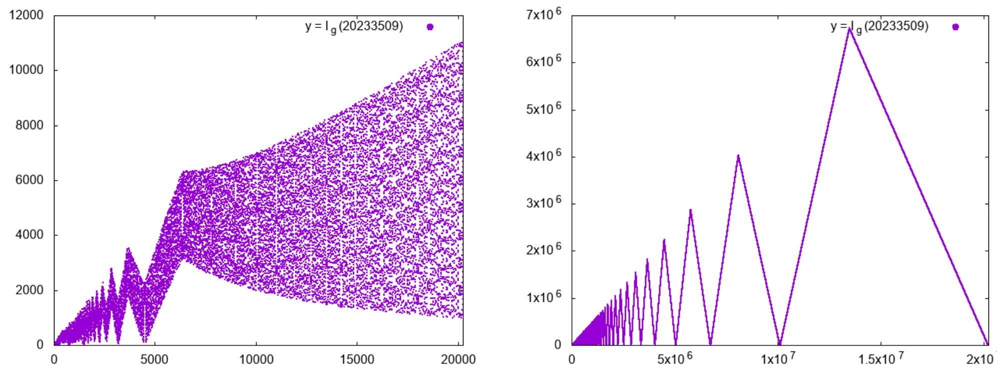

FIGURE 2. Plots of $y = \ell_g(20233509)$ as a function of g.

TABLE 1. First few values of $\lambda_p(k)$ for primes $p < 30$.

k	2	3	5	7	11	13	17	19	23	29
1	1	1	1	1	1	1	1	1	1	1
2	3	2	2	2	2	2	2	2	2	2
3	11	5	3	3	3	3	3	3	3	3
4	43	14	8	4	4	4	4	4	4	4
5	171	41	13	11	5	5	5	5	5	5
6	683	122	38	18	6	6	6	6	6	6
7	2731	365	63	25	17	7	7	7	7	7
8	10923	1094	188	74	28	20	8	8	8	8
9	43691	3281	313	123	39	33	9	9	9	9
10	174763	9842	938	172	50	46	26	10	10	10
11	699051	29525	1563	515	61	59	43	29	11	11
12	2796203	88574	4688	858	182	72	60	48	12	12
13	11184811	265721	7813	1201	303	85	77	67	35	13
14	44739243	797162	23438	3602	424	254	94	86	58	14
15	178956971	2391485	39063	6003	545	423	111	105	81	15
16	715827883	7174454	117188	8404	666	592	128	124	104	44
17	2863311531	21523361	195313	25211	1997	761	145	143	127	73
18	11453246123	64570082	585938	42018	3328	930	434	162	150	102
19	45812984491	193710245	976563	58825	4659	1099	723	181	173	131
20	183251937963	581130734	2929688	176474	5990	3296	1012	542	196	160

Figure 3 shows the integers less than 10,000 and their 19-length, together with the graph of $y = \lambda_{19}(x)$.

The following two theorems give an explicit formula for λ_g. First, we need a preliminary lemma that relates the digits in the minimal g-adic expansion of an integer to its size.

Lemma 3.5. *Let $g > 1$ be an integer. Let m and n be distinct integers with minimal g-adic expansions $[n]_g = [n_0, n_1, \ldots, n_N]$ and $[m]_g = [m_0, m_1, \ldots, m_M]$. Set $n_i = 0$ for $i > N$ and $m_i = 0$ for $i > M$. Let $t \geq 0$ be the largest integer such that $n_t \neq m_t$. Then $n > m$ if and only if $n_t > m_t$.*

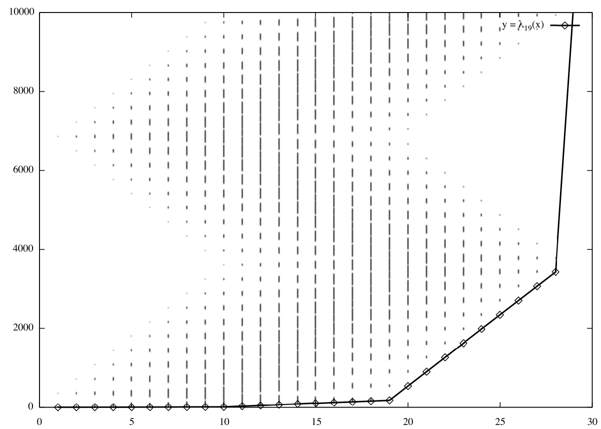

FIGURE 3. The integers up to 10,000 whose 19-lengths are x are shown together with the graph of $y = \lambda_{19}(x)$.

Proof. By subtracting $s = \sum_{i=t+1}^{\infty} n_i g^i$, we can assume without loss of generality that $n = \sum_{i=0}^{t} n_i g^i$ and $m = \sum_{i=0}^{t} m_i g^i$, so that $N = M = t$. Since $-n = \sum_{i=0}^{N}(-n_i)g^i$ and similarly for m, it suffices consider positive integers m and n. Relabel if necessary to assume without loss of generality that $n_N > m_N$. Then

$$n - m = \sum_{i=0}^{N}(n_i - m_i)g^i \geq g^N + \sum_{i=0}^{N-1}(n_i - m_i)g^i.$$

Thus it suffices to show

$$\sum_{i=0}^{N-1}(n_i - m_i)g^i < g^N. \tag{1}$$

Let $n' = \sum_{i=0}^{N-1} n_i g^i$, and let $m' = \sum_{i=0}^{N-1} m_i g^i$. The left side of (1) is maximized when n' is positive and as large as possible, and m' is negative and as small as possible, in which case we claim that $n_i - m_i \leq g - 1$. Thus, it suffices to show (1) in this case.

Suppose g is odd. Let $b = \frac{g-1}{2}$. Theorem 3.1 implies m_i and n_i are in $\{0, \pm 1, \ldots, \pm b\}$, so

$$n_i - m_i \leq b + b = g - 1. \tag{2}$$

Suppose g is even. Let $b = \frac{g}{2}$. Theorem 3.2 implies m_i and n_i are in $\{0, \pm 1, \ldots, \pm b\}$. Furthermore, if a digit $|\epsilon_i| = b$, then $|\epsilon_{i+1}| < b$ and $\epsilon_i \epsilon_{i+1} \geq 0$. Note that $m'_{N-1} \neq -b$, since $m > 0$. It follows that the smallest m' can be is when the minimal g-adic expansion of m' alternates $-b$ and $-(b-1)$; the largest n' can be is when the minimal g-adic expansion of n' alternates between $b-1$ and b. Thus

$$[m']_g \geq [\ldots, -b, -(b-1)] \quad \text{and} \quad [n']_g \leq [\ldots, b-1, b],$$

so

$$n_i - m_i \leq b + (b-1) = g - 1. \tag{3}$$

Thus, we have

$$\sum_{i=0}^{N-1}(n_i - m_i)g^i \leq \sum_{i=0}^{N-1}(g-1)g^i = (g-1)\sum_{i=0}^{N-1} g^i = (g-1)\left(\frac{g^N - 1}{g - 1}\right) = g^N - 1,$$

so (1) follows. ☐

Theorem 3.6. *Let $g > 1$ be an odd integer, and let $k > 0$ be an integer. Let $q = \lfloor \frac{2k}{g-1} \rfloor$, and let $r = k \bmod \frac{g-1}{2}$ so that $k = q(\frac{g-1}{2}) + r$. Let*

$$A = \begin{cases} \frac{g-1}{2} & \text{if } r = 0, \\ -\left(\frac{g-1}{2}\right) & \text{otherwise;} \end{cases} \qquad B = \begin{cases} 0 & \text{if } r = 0, \\ r & \text{otherwise;} \end{cases}$$

Then

$$\lambda_g(k) = \frac{1 - g^{q-1}}{2} + Ag^{q-1} + Bg^q.$$

Proof. Let $b = \frac{g-1}{2}$, and let $n = \frac{1-g^{q-1}}{2} + Ag^{q-1} + Bg^q$. A straightforward computation shows that the minimal g-adic expansion of n is given by

$$[n]_g = \begin{cases} \underbrace{[-b, -b, \dots, -b, b]}_{q \text{ digits}} & \text{if } r = 0, \\ \underbrace{[-b, -b, \dots, -b, r]}_{q+1 \text{ digits}} & \text{otherwise.} \end{cases} \tag{4}$$

First, we show $\ell_g(n) = k$. There are two cases to consider. If $r = 0$, we have $q = \lfloor \frac{2k}{g-1} \rfloor = \frac{2k}{g-1}$. It follows that

$$\ell_g(n) = bq = \left(\frac{g-1}{2}\right)\left(\frac{2k}{g-1}\right) = k,$$

as desired. If $r \neq 0$, then

$$\ell_g(n) = bq + r = k,$$

by construction.

Finally, we show that n is the smallest positive integer with this property. Suppose $m < n$ is a positive integer. Let $[m]_g = [m_0, m_1, \dots, m_M]$ be the minimal g-adic expansion of m, and let $[n]_g = [n_0, n_1, \dots, n_N]$ be the minimal g-adic expansion given in (4). Set $n_i = 0$ for $i > N$, and set $m_i = 0$ for $i > M$. Let $t \geq 0$ be the largest integer such that $m_t \neq n_t$. By Lemma 3.5, we must have $m_t < n_t$. By Theorem 3.1, we have $|m_j| \leq b$, for all j. Since $n_j = -b$ for $0 \leq j \leq N-1$, we cannot have $t \leq N-1$. Since $0 < m < n$, we must have $M \leq N$, and hence $t = N$. Thus $m_N < n_N$. Furthermore, $m_j = n_j = 0$ for $j > N$. It follows that $\ell_g(m) < \ell_g(n)$. ☐

Theorem 3.7. *Let $g > 1$ be an even integer, and let $k > 0$ be an integer. Let $r = k \bmod g-1$. Define integers q, A, and B by*

$$q = \begin{cases} \lfloor \frac{k}{g-1} \rfloor - 1 & \text{if } r = 0, \\ \lfloor \frac{k}{g-1} \rfloor & \text{otherwise;} \end{cases}$$

$$A = \begin{cases} \frac{g}{2} & \text{if } r = 0 \text{ or } r > \frac{g}{2}, \\ r & \text{otherwise;} \end{cases}$$

$$B = \begin{cases} \frac{g}{2} - 1 & \text{if } r = 0, \\ r - \frac{g}{2} & \text{if } r > \frac{g}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\lambda_g(k) = \frac{g(1-g^{2q})}{2(1+g)} + Ag^{2q} + Bg^{2q+1}.$$

Proof. Let $b = \frac{g}{2}$, and let

$$n = \frac{g(1-g^{2q})}{2(1+g)} + Ag^{2q} + Bg^{2q+1}.$$

A straightforward computation shows

$$[n]_g = \begin{cases} \underbrace{[-b, -(b-1), -b, -(b-1), \dots, -b, -(b-1), b, b-1]}_{2q+2 \text{ digits}} & \text{if } r = 0, \\ \underbrace{[-b, -(b-1), -b, -(b-1), \dots, -b, -(b-1), b, r-b]}_{2q+2 \text{ digits}} & \text{if } r > b, \\ \underbrace{[-b, -(b-1), -b, -(b-1), \dots, -b, -(b-1), r]}_{2q+1 \text{ digits}} & \text{otherwise.} \end{cases} \tag{5}$$

First, we show $\ell_g(n) = k$. There are three cases to consider. Suppose $r = 0$. Then $g-1$ divides k, so

$$q = \left\lfloor \frac{k}{g-1} \right\rfloor - 1 = \frac{k}{g-1} - 1.$$

Then

$$\begin{aligned} \ell_g(n) &= q(2b-1) + b + (b-1) \\ &= \left(\frac{k}{g-1} - 1 \right)(g-1) + \frac{g}{2} + \left(\frac{g}{2} - 1 \right) \\ &= k. \end{aligned}$$

If $r > \frac{g}{2}$, then

$$\begin{aligned} \ell_g(n) &= q(2b-1) + b + (r-b) \\ &= \left\lfloor \frac{k}{g-1} \right\rfloor (g-1) + r \\ &= k. \end{aligned}$$

If $1 \leq r \leq b$, then

$$\begin{aligned} \ell_g(n) &= q(2b-1) + r \\ &= \left\lfloor \frac{k}{g-1} \right\rfloor (g-1) + r \\ &= k. \end{aligned}$$

Finally, we show that n is the smallest positive integer with this property. Suppose $m < n$ is a positive integer. We need to show $\ell_g(m) < \ell_g(n)$. Let $[m]_g = [m_0, m_1, \dots, m_M]$ be the minimal g-adic expansion of m, and let $[n]_g = [n_0, n_1, \dots, n_N]$ be the minimal g-adic expansion given in (5). Let $t \geq 0$ be the largest integer such that $m_t \neq n_t$. By Lemma 3.5, we must have $m_t < n_t$.

First we show that $t \geq 2q$. Note that if $t < 2q$, then t cannot be even, since $n_t = -b$ for even t in this range and Theorem 3.2 implies $|m_t| \leq b$. Furthermore, t cannot be odd, since in that case $m_t = -b$. Then by Theorem 3.2, we have $|m_{t+1}| < b$ and $m_{t+1} \leq 0$. We have $n_{t+1} = -b$ for odd $t < 2q - 1$ and $n_{t+1} > 0$ for $t = 2q - 1$, so this cannot occur since $m_{t+1} = n_{t+1}$ by the definition of t. Thus $t \geq 2q$, as desired.

We have

$$\ell_g(m) = \left(\sum_{i=0}^{t-1} |m_i|\right) + |m_t| + \left(\sum_{i=t+1}^{\infty} |m_i|\right).$$

We have $\left(\sum_{i=t+1}^{\infty} |m_i|\right) = \left(\sum_{i=t+1}^{\infty} |n_i|\right)$ by the definition of t. Since $m < n$, we have $M \leq N$ so $t \leq N$. Thus there are two cases to consider.

First suppose $t = 2q$. Then we have

$$\left(\sum_{i=0}^{t-1} |m_i|\right) \leq (2b-1)q = \left(\sum_{i=0}^{t-1} |n_i|\right)$$

from (5) and Theorem 3.2. Thus it suffices to show $|m_t| < |n_t|$. If $0 < r <\leq b$, then $t = 2q = N$ so $m_t \geq 0$ since $m > 0$, and $m_t < n_t$ from Lemma 3.5. If $r = 0$ or $r > b$, then $n_t = b$. Note that $m_t \neq -b$, since otherwise $m_{t+1} \leq 0$ from Theorem 3.2 but $m_{t+1} = n_{t+1} = n_N > 0$ from (5). It follows that $|m_t| < |n_t|$, as desired.

Finally, suppose $t = 2q + 1$. Then necessarily $r = 0$ or $r > b$ and $t = 2q + 1 = N$. It follows that $m_t \geq 0$ since $m > 0$, and $m_t < n_t$ from Lemma 3.5 so $|m_t| < |n_t|$. We have

$$\left(\sum_{i=0}^{t-1} |m_i|\right) \leq (2b-1)q + b = \left(\sum_{i=0}^{t-1} |n_i|\right)$$

from (5) and Theorem 3.2, so $\ell_g(m) < \ell_g(n)$, as desired. □

4. Metric properties of C_P

Let P be a set of positive integers. Let $C_P = \Gamma(\mathbb{Z}, S_P)$ denote the Cayley graph of $\mathbb{Z}$ with the generating set

$$S_P = \bigcup_{a \in P} \{\pm a^i \mid i \in \mathbb{Z}_{\geq 0}\}.$$

We give C_P the edge-length metric d_{S_P}, and use $\ell_P(n)$ to denote the P-length of n in the metric d_{S_P}, i.e., $\ell_P(n) = d_{S_P}(0, n)$. The P-length function is much more subtle when $\#P > 1$.

Question 4.1. Let P be a subset of primes. Let $\lambda_P(h)$ denote the smallest positive integer of P-length h in C_P. Compute the function $\lambda_P(h)$.

There are partial results addressing Question 4.1 when $\#P < \infty$. Hadju and Tijdeman [6] prove that $\exp(ck) < \lambda_P(k) < \exp((k \log k)^C)$, with some constant c depending on P and an absolute constant C.

Nathanson [8] gives a class of generating sets for $\mathbb{Z}$ whose arithmetic diameters are infinite.

Theorem 4.2 ([8, Theorem 5]). *If P is a finite set of positive integers, then C_P has infinite diameter.*

On the other hand, for infinite P the diameter of C_P may be finite. The *ternary Goldbach conjecture* states that every odd integer n greater than 5 can be written as the sum of three primes. Helfgott's proof [7, Main Theorem] of this implies if $\mathcal{P}$ is the set of all primes, then $C_{\mathcal{P}}$ is at most 4.

Theorem 4.3. *Let $\mathcal{P}$ be the set of all primes. The diameter of $C_{\mathcal{P}}$ is 3 or 4.*

Proof. It is easy to see that $\ell_{\mathcal{P}}(n) = 1$ for $n \in \{1, 2, 3, 4, 5\}$. Helfgott [7, Main Theorem] proves that every odd integer greater than 5 can be written as the sum of three primes. Since every even integer greater than 4 can be expressed as 1 less than an odd integer greater than 5, we have that $\ell_{\mathcal{P}}(n) \leq 4$ for all $n \in \mathbb{Z}$.

Since not every integer is a prime power, the diameter of $C_{\mathcal{P}}$ is at least 2. To show that the diameter is not 2, it suffices to produce an integer that is not a prime power and cannot be expressed as the sum or difference of prime powers, where the prime power $p^0 = 1$ is allowed. Such integers are surprisingly hard to find. First note that the Goldbach conjecture asserts that every even integer greater than 2 can be expressed as the sum of two primes. This has been computationally verified integers less than $4 \cdot 10^{18}$ [10]. It follows that $\ell_{\mathcal{P}}(n) \leq 2$ for even integers $n < 4 \cdot 10^{18}$. Thus a search for an integer of P-length 3 should be restricted to odd integers. An odd integer M is P-length 3 if

(1) M is not prime power;
(2) $|M \pm 2^n|$ is not prime power for all $n \geq 0$.

Cohen and Selfridge [4, Theorem 2] use covering congruences to prove the existence of an infinite family of integers M satisfying item (2) and give an explicit 94-digit example of such an integer. Sun [11] adapts their work to produce a much smaller example. Specifically, let

$$M = 47867742232066880047611079, \quad \text{and let} \quad N = 66483084961588510124010691590.$$

Sun proves that if $x \equiv M \bmod N$, then x is not of the form $|p^a \pm q^b|$ for any primes p, q and nonnegative integers a, b[1]. We use Atkin and Morain's ECPP (Elliptic Curve Primality Proving) method [2] implemented by Morain in Magma [3] to look in this congruence class for an element that is provably not a prime power. We find that M and $M + N$ are prime, but

$$\begin{aligned} M + 2N &= 133014037665409087128068994259 \\ &= 23 \cdot 299723 \cdot 19295212676140402555471 \end{aligned}$$

is not a prime power. Thus $\ell_{\mathcal{P}}(M + 2N) = 3$, and the result follows. □

Remark 4.4. Assuming Goldbach's conjecture, the diameter of $C_{\mathcal{P}}$ is 3.

It is still an open problem to find the smallest integer n that is not of the form $|p^a \pm q^b|$, for any primes p, q and nonnegative integers a, b [5, A19]. Explicit computations [4, 11] show that the smallest such integer must be larger than 2^{25}. Such elements, if not prime powers, would have $\mathcal{P}$-length 3. We have extended slightly their computation and confirmed that $\ell_{\mathcal{P}}(n) < 3$ for all $n < 58{,}164{,}433 \approx 2^{25.79}$. For

$$n = 58164433 = 4889 \cdot 11897,$$

we could not show $\ell_{\mathcal{P}}(n) = 2$. It is possible that this integer is the smallest positive integer of $\mathcal{P}$-length 3.

Corollary 4.5. *Let P be a subset of the natural numbers containing all but finitely many primes. Then, C_P has finite diameter.*

[1] The modulus N given by Sun [11] is incorrectly written as 66483034025018711639862527490.

Proof. We need to show that $\ell_P(n)$ is bounded for all n sufficiently large. It is enough to consider the case $P = \mathcal{P} \setminus S$, where S is finite. Let $R = \max_{p\in S}\{\ell_P(p)\}$.

First note that if p is a prime in S, then $\ell_P(p) \leq R$. If p is a prime not in S, then $\ell_P(p) = 1$. Thus $\ell_P(p) \leq R$ for any prime p.

Since every even integer is one less than an odd integer, it suffices to show that $\ell_P(n) \leq 3R$ for every positive odd integer n that is sufficiently large. By the ternary Goldbach conjecture, every odd integer $n > 5$ can be expressed as the sum of three primes. Let $n = p + q + r > 5$ be an odd integer for some primes p, q, r. Then

$$\ell_P(n) \leq \ell_P(p) + \ell_P(q) + \ell_P(r) \leq 3R,$$

and the result follows. □

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